A RENAISSANCE OF GEOMETRIC CONSTRUCTIONS

Richard L. Francis

Abstract. “A Renaissance of Geometric Constructions” emphasizes the 200th anniversary of Gauss's 1796 heptadecagon construction discovery. Its content is set in a broad historical perspective. The modern era awakening of interest in the ancient art of Euclidean constructions, symbolized by the Gaussian breakthrough, is seen in the light of historical impact, not only on the nineteenth century but twentieth century mathematics as well.

1. Introduction. In 1796, a major geometric advancement occurred, a breakthrough whose 200th anniversary now appears on the calendar of great events. Gauss's famous discovery of a method of constructing the regular 17–sided polygon answered a critical question of long standing. It too was a landmark event which typified, more perhaps than any other activity of its kind, a rebirth of interest in the fluctuating challenges of geometric constructions.

The golden era of ancient demonstrative mathematics placed considerable emphasis on the concept of locus and the allied topics of straightedge and compass constructions. This focus, declining somewhat at the time of the commentators of the Later Greek Era, disappeared almost entirely in western culture with the advent of the Dark Ages. For centuries, the subject lay dormant except for occasional findings in the Near Eastern settings (e.g., as by Abul–Wefa of the tenth century). On the European continent, little formal interest attached to the elementary Euclidean constructions which are now so notably a part of introductory geometry courses (segment and angle bisections, constructions of perpendiculars and parallels, etc.). A gradual resurfacing of classical mathematical masterpieces in the fifteenth and sixteenth century west created a climate favorable to varied encounters with long lost topics. Seventeenth century mathematicians (such as Gregoire de Saint Vincent and Rene Descartes) were to serve as forerunners in this renaissance of constructions but Carl Friedrich Gauss (1777–1855) ultimately proved the outstanding figure.

Gauss was not alone in this renewed area of intense construction interest. Others, including the Italian mathematician Lorenzo Maschernoi (1750–1800), had discovered that all Euclidean constructions of a point–by–point nature could be effected with the unmarked straightedge alone. Such a discovery appeared in his 1797 publication *Geometria del Compasso* (a finding from a century earlier in the scarcely known works of George Mohr). Likewise, mathematicians of such stature as Jean–Victor Poncelet (1788–1867) and Jacob Steiner (1798–1867) were to follow in the footsteps of Gauss.

2. The Declining Years of the Eighteenth Century. The middle years of the last decade of the eighteenth century identified the beginning of a remarkable chain of mathematical discoveries by the young Gauss. As early as 1795, Gauss had given evidence of strong feelings on the subject of the Law of Quadratic Reciprocity. Such probings were in time to lead to some contention between the upcoming mathematician and his older rival Legendre. In the following year, Gauss made his discovery of the detailed steps of the regular heptadecagon construction. It was a decided move forward, even a quantum leap, considering the construction limitations of the ancients.

Prior to 1796, the known constructibility cases of odd–sided regular polygons were restricted to those of 3, 5, and 15 sides. Building on insights stemming from much earlier achievements, Gauss proved successful in finding the subtle steps which led to his first major mathematical contribution, that of constructing the regular heptadecagon with the allowable instruments of the unmarked straightedge and the compass. This achievement of 1796, occurring at the University of Göttingen, was to prove a forerunner to the productive earlier years of the nineteenth century.

Still another cornerstone of mathematics was a consequence of the works of Gauss in the last decade of the eighteenth century. Unyielding to Euler and others who surmised its truth, the Fundamental Theorem of Algebra lay at the cutting edge of mathematics. In his doctoral dissertation at the University of Helmstedt, Gauss established that every polynomial equation has at least one complex root. The year was 1799. Various corollaries immediately emerged from this landmark result called the Fundamental Theorem of Algebra. In time, Gauss was to prove the theorem four ways. As none of these proofs were purely algebraic in nature, evidence was given of Gauss's impressive insight and his ultimate acknowledgment as a mathematical universalist. Other interests demanded his time, including astronomy and pioneering attempts at what is today called non–Euclidean geometry.

Significantly, Gauss questioned the parallel postulate of Euclid as early as the year 1794.

3. The Gaussian Construction Standard. In his penetrating treatise, *Disquisitiones Arithmeticae*, Gauss returns to the subject of geometric constructions, the very subject which pointed him career–wise in the direction of mathematics. This fabulous collection ranged over the theory of congruences, quadratic residues, quadratic forms, and algebraic equations. It included in its later sections a regular polygonal construction standard. The standard, of an existential kind, rested on insights of number theory which had been formulated in the earlier years of the modern era.

In *Disquisitiones Arithmeticae*, Gauss demonstrated that an odd–sided regular polygon is constructible if its number of sides is a Fermat prime (one of the form 2^k+1) or the product of distinct Fermat primes. Hence, not only were the regular 3–sided and 5–sided polygons constructible (as known to the ancient Greeks) but also the 17–sided regular polygon. Interestingly, the only Fermat primes known today are the numbers 3, 5, 17, 257, and 65537. Accordingly, the regular polygon of (3)(5)(17)(257)(65537) sides is constructible. It contains the largest known number of sides for which an odd–sided regular polygon can be constructed by use of the Euclidean instruments.

The above account expresses a sufficient condition of constructibility. Though the condition is also necessary in the odd–sided case, there is serious reservation among mathematicians as to Gauss's role in the verification. The consensus is that Gauss did not produce a proof of the necessity of the standard in spite of his careful consideration of it. Only by application of such a converse can it be said emphatically that the regular 7–sided polygon is not constructible (a problem of some challenge to the ancients and the first of the regular polygons to be so described). Proof of the necessity of such a condition is ordinarily attributed to Pierre Laurent Wantzel (1814–1848). The overall criterion which contains both sufficiency and necessity is sometimes referred to as the Gaussian Constructibility Standard Extended. It may be expressed concisely by use of Euler's famous totient function (and concerns even–sided as well as odd–sided cases). That is, a regular polygon of n sides ($n > 2$) is constructible if and only if $\phi(n)$ is a power of 2.

It is interesting to note that the degree, which is the fundamental unit of sexagesimal angle measure, is not constructible. For, if it were, the 40–degree angle would be constructible (by repetition). As this is the central angle of a regular nonagon, the nonagon's constructibility would violate the Gaussian Standard Extended. That is, 9 is the product of non–distinct Fermat primes. By a similar analysis, neither is the minute nor the second constructible.

4. Beyond the Heptadecagon. Suppose the construction steps are available in the case for the regular triangle and the regular heptadecagon. How then might a regular polygon of (3)(17) sides be constructed? Begin with the linear Diophantine equation $3x + 17y = 1$ which may be solved (by the Euclidean Algorithm) so as to yield a solution (say, $x = -11$ and $y = 2$). Such an equation is solvable in integers as the coefficients are distinct primes and thus have a greatest common divisor of 1. Writing the equation as

$$\frac{x}{17} + \frac{y}{3} = \frac{1}{51}$$

and multiplying by 360 (degrees) gives

$$360 \cdot \left[\frac{x}{17}\right] + 360 \cdot \left[\frac{y}{3}\right] = 360 \cdot \left[\frac{1}{51}\right].$$

Letting $x = -11$ and $y = 2$, it follows that the subtracting of 11 times the central angle of the regular heptadecagon from two times the central angle of the regular triangle produces the central angle of a regular 51–sided polygon. Note that

$$2(120) - 11\left(21\frac{3}{17}\right) = 240 - 232\frac{16}{17} = 7\frac{1}{17} = \frac{360}{51}.$$

The reader may wish to use this technique to construct a regular 85–sided regular polygon. It is interesting to compare this composite construction with the ancient Greek method of constructing the regular pentadecagon.

An actual detailing of the construction steps for a regular 257–gon is attributed to the mathematician R. J. Richelot (1808–1875) in 1832. The Hermes construction of the regular 65537–gon (around the year 1894) rounded out historically the regular polygon constructibility steps for all known Fermat primes. Should these detailed processes be available to the geometer, how then might a regular polygon of (3)(5)(17)(257)(65537) sides be constructed? Paralleling the development above, assume that all regular polygon construction steps based on products of two, three, and four distinct Fermat primes are available. Now begin with the equation

$$3x + 5y + 17z + 257v + 65537w = 1,$$

which is clearly solvable in integers. Among its solutions are the numbers $x = -21940$, $y = 2$, $z = 1$, $v = 1$, and $w = 1$. The process of dividing all members of the first equation by (3)(5)(17)(257)(65537) and then multiplying each term by 360 (degrees) yields the central angle of the desired regular polygon. That is, if the central angles associated with the left member are respectively A, B, C, D, and E, then $2B + C + D + E - 21940A$ is the central angle of a regular polygon of (3)(5)(17)(257)(65537) sides. The description of the steps of constructing this regular 4,294,967,295–sided polygon thus becomes a relatively simple matter. It should be noted that if n is the cardinality of the set of Fermat primes, there are but $2^n - 1$ constructible regular polygons of an odd number of sides.

5. The Three Famous Problems of Antiquity. Problems of long standing motivated much of the mathematical activity of this time period called the Renaissance of Geometric Constructions. Even as the ancients expressed curiosity about extended regular polygon constructions, they did the same concerning variations on other familiar constructions. Hence, angle bisection, square duplication, and polygon squaring led to the counterpart consideration of angle trisection, cube duplication, and circle squaring. These famous problems of antiquity thus made their inevitable reappearance at the time of a rekindling of interest in the broad area of geometric constructions. The onslaught of this renaissance was to make use of the more powerful tools of analysis.

If in fact Gauss had proved both the necessity and sufficiency of the regular polygonal constructibility standard, then he likewise must be acclaimed as the one who demonstrated the impossibility of general angle trisection. As by such a standard, the regular nonagon is not constructible, neither is its central angle of 40 degrees. This clearly rules out the possibility of trisecting the constructible 120 degree angle. Thus, angle trisection impossibility (in general) could be seen as an instant corollary of a much broader theorem. Gauss and others would surely have heralded the fact if it had been then and there discovered. It and possibly Fermat's "Theorem" were among the most famous of the problems of the early nineteenth century.

As noted, Wantzel is generally accorded the honor of angle trisection resolution. His disposition necessitated a powerful theorem from the theory of equations, namely, if an algebraic equation of degree 3 has no rational roots, then none of its roots are constructible. Building on the identity

$$\cos 3y = 4\cos^3 y - 3\cos y$$

and replacing y by $20°$ yields $\cos 60° = 4\cos^3 20° - 3\cos 20°$. If $\cos 20°$ is replaced by x, then

$$\frac{1}{2} = 4x^3 - 3x \quad \text{or} \quad 8x^3 - 6x - 1 = 0.$$

As the preceding equation has no rational roots, none of its roots (including $\cos 20°$) are constructible. The non–constructibility of $\cos 20°$ implies the non-constructibility of the $20°$ angle. Accordingly the constructible $60°$ angle cannot be trisected. The same theorem of Wantzel resolves the second famous problem of antiquity, the duplication of a cube. This finding stems from the fact that the cube root of 2 is a non–constructible length. That is, $x^3 = 2$ has no rational roots. Hence, none of its roots, including $\sqrt[3]{2}$, are constructible. These resolutions of the first two of the famous problems of antiquity occurred in the year 1837.

The third problem of antiquity, the squaring of a circle, was resolved in 1882 by the German mathematician C. F. Lindemann (1852–1939). His proof was essentially that of establishing the transcendence of π and rests on the then known fact that no transcendental length is constructible.

All in all, the Renaissance of Geometric Constructions, beginning late in the eighteenth century, resulted in far–reaching accomplishments in a period of less than nine decades. These insightful breakthroughs encompassed among other things the complete characterization of regular polygon constructibility as well as rigorous demonstration of impossibility in the case for the three famous problems of antiquity.

6. Asymptotic Constructions. The construction advancements of Gauss, Mascheroni, and Wantzel were of a monumental kind. Yet the subject was far from exhausted. Mathematicians, bending the rules of construction, acquired a new zeal and enthusiasm. In the spirit of Archimedes, who "trisected" a general angle by a compromising of the unmarked straightedge restriction, many, professional and amateur alike, were to follow.

Many of the compromises of the Euclidean conditions involved the use of curves other than the line and the circle (such as the quadratrix of Hippias, the trisectrix of Maclaurin, the conchoid of Nicomedes, or the spiral of Archimedes). A rarely mentioned Euclidean restriction is that all constructions must be performed in a finite number of steps. Should this restriction be set aside, resulting constructions are thus of an asymptotic kind. For example, angle quadrisection is possible by repeated bisecting. Suppose then that the angle θ to be trisected is approached by repeated quadrisecting but in an endless manner. The geometric series which summarizes the steps is given by

$$\frac{\theta}{4}+\frac{\theta}{16}+\frac{\theta}{64}+\frac{\theta}{256}+\cdots .$$

It converges to $\frac{\theta}{3}$, and is thus the desired construction.

Similarly, the duplication of a cube necessitates the construction of the cube root of 2 (which is an impossible Euclidean construction by Wantzel's Theorem). However, the construction of the square root of a given number and consequently, the fourth root, are allowable. Consider the infinite product

$$2^{\frac{1}{4}} \cdot 2^{\frac{1}{16}} \cdot 2^{\frac{1}{64}} \cdot 2^{\frac{1}{256}} \cdot \;\ldots$$

which results in

$$2^{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots}$$

that is, the cube root of 2. As in the case for angle trisection, an asymptotic construction results in the desired duplication.

The squaring of a circle, requiring the construction of π, can be accomplished in various ways asymptotically. Among these approaches is utilization of a series, today called the Gregory Series, given below.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

By an endless construction of the various segments and by an appropriate addition and subtraction, the construction of π is performed. The circle can thus be squared asymptotically.

Interestingly, all regular polygons are constructible if this construction type is permitted. For example, to construct the regular heptagon, simply use the infinite series

$$\frac{\theta}{8} + \frac{\theta}{64} + \frac{\theta}{512} + \frac{\theta}{4096} + \cdots$$

which converges to $\frac{\theta}{7}$. This of course permits the dividing of $360°$ into seven "equal" parts. Such a process is easily modified so as to account for the regular n–sided polygon.

7. Extensions. Related milestones of twentieth century mathematics touched on the subject of construction comparisons and various standards of elegance. Emile Leomoine is remembered for a contributory role in the overall science of construction evaluations along the lines of "good, better, and best." Such a science, called "geometrography," involves more than a mere counting of construction steps as in the dividing of a line segment into a million "equal" parts or constructing a regular 257–gon. The latter is surely the more challenging in spite of its lesser number of steps and clearly identifies the difficulties inherent in making geometric judgments that are seemingly of a subjective nature.

Among the more notable marks of modern mathematics are abstraction and generalization. Constructions have fared well in this latter area of modern day pursuit as have other late discoveries of Euclidean geometry (such as the Steiner–Lehmus–Terquem Theorem of the mid–nineteenth century or the Morley Triangle Theorem of the early twentieth century.) The famous problems of antiquity illustrate this well.

The angle multisection problem in the context of strict Euclidean construction is completely resolved. The only multisections possible are those involving powers of two (bisection and repeated bisection). Note for example that if pentasection were possible then the 360–degree angle could be pentasected and the resulting angle then divided by 5. This implied construction of the regular 25–gon violates the Gaussian standard because of the repetition of Fermat prime factors. By a similar argument, utilizing the Gaussian Constructibility Standard Extended, all general angle multisections are precluded except those involving powers of two. Likewise, integral multipliers of the volume of a given cube must themselves be exact cubes. To triplicate a cube for example requires the construction of the cube root of 3 (which is impossible by Wantzel's Theorem). Moreover, as any polygon can be squared, it follows that no polygon, regular or otherwise, can be constructed having the same area as a given circle.

Variations, many of which preceded or clustered around the remarkable year 1796, easily took the mathematician into the realm of other figure types. Among these was the construction of a sixth point on a conic section if five of its points are known. Or the construction of the Pascal line of a hexagon (re-entrant or otherwise) if inscribed in a circle. Or the construction of the Desargues line of two centrally perspective triangles; and more!

Even as the discovery of the calculus became the historical dividing line between elementary and modern mathematics, so too did Gauss's regular heptadecagon discovery symbolize a renewed interest in the ancient art of geometric constructions. The symbolism is reinforced by Gauss's perception of the inscribed regular seventeen–sided polygon as a fitting epitaph of his life's work. Such a Renaissance of Geometric Constructions was to transcend by far its counterparts in antiquity. Its modern and renascent pursuits and resulting advancements were to give evidence of the broader scope of the discipline and a unity of mathematical thought undreamed of by the ancient world.

References

1. W. K. Buhler, *Gauss: A Biographical Study.* New York: Springer–Verlag, 1981.

2. H. W. Eves, *An Introduction to the History of Mathematics* Philadelphia: Saunders College Publishing, 1990.

3. H. W. Eves, *A Survey of Geometry*, Vol. 1, Boston: Allyn and Bacon, Inc., 1963.

4. R. L. Francis, "A Note on Angle Construction," *The College Mathematics Journal*, 9, (1978) 75–80.

5. R. L. Francis, "Did Gauss Discover That Too?" *Mathematics Teacher*, 79, (1986) 288–293.

6. R. L. Francis, "Just How Impossible Is It?" *Journal of Recreational Mathematics*, 20, (1988) 241–248.

7. R. L. Francis, "The Lost Chord," *Consortium for Mathematics and Its Applications*, 38, (1991) 10–11.

8. C. F. Gauss, *Disquisitiones Arithmeticae.* Clark, Arthur A., trans. New Haven: Yale University Press, 1966.

9. N. D. Kazarinoff, "On Who First Proved the Impossibility of Constructing Regular Polygons with Ruler and Compass Alone," *American Mathematical Monthly*, 75, (1968) 647.

Richard L. Francis
Southeast Missouri State University
Department of Mathematics
Cape Girardeau, MO 63701

Figure 1.

A Regular Heptadecagon Inscribed in a Circle

Figure 2.

Archimedean Method of Angle Trisection