THE CIRCUMCIRCLE AND THE LINE AT INFINITY

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1. Introduction. For a pleasant surprise, pick up the nearest pencil and carry out a little construction: begin with a circle, and mark any three points on it. Label them A, B, C, and draw their triangle. Starting at any fourth point, X, on your circle, draw line AX and reflect it about the bisector of angle A, obtaining a line L_1. Then reflect BX about the B-bisector and CX about the C-bisector to obtain lines L_2 and L_3. The surprise is that the three lines are parallel, meeting in a point, X^{-1}, infinitely far from $\triangle ABC$. The set of all such points is called the *line at infinity*, denoted $\mathcal{L}^\infty$, and the plane of $\triangle ABC$ together with $\mathcal{L}^\infty$ is the *extended plane* of $\triangle ABC$.

If the initial point X is not on your circle (i.e., the *circumcircle*, Γ, of $\triangle ABC$) and not on a sideline of $\triangle ABC$, then the three reflected lines are not parallel. Try out a couple of X with your pencil, one inside Γ and the other outside. You will discover another pleasing fact: the three lines L_1, L_2, L_3 concur in a single point, again written as X^{-1} and called the *isogonal conjugate* of X. Of course, $(X^{-1})^{-1} = X$.

We define six more isogonal conjugates by the equations $A^{-1} = BC \cap \mathcal{L}^\infty$, $B^{-1} = CA \cap \mathcal{L}^\infty$, $C^{-1} = AB \cap \mathcal{L}^\infty$, $(A^{-1})^{-1} = A$, $(B^{-1})^{-1} = B$, $(C^{-1})^{-1} = C$, so that now we may describe $\mathcal{L}^\infty$ as follows: (i) each of its points is essentially a family of parallel lines, and (ii) $\Gamma^{-1} = \mathcal{L}^\infty$. Here, of course, we are referring to isogonal conjugation as a mapping of one set onto another.

2. Antipedal Lines. Figure 1 shows antipedal lines for a triangle ABC, and Figure 2 represents these antipedal lines by short segments. In this second figure, if you start at any X on Γ and let θ_0 be the angle ($\leq \pi/2$) between the antipedal line XX^{-1} and the tangent to Γ at X, you will encounter two other points on Γ for which the corresponding angle θ equals the original θ_0. In fact, as is clear from the constant rate of change of θ as X moves around Γ, the three points form an equilateral triangle. Later we shall discuss two of these triangles: the *circumtangential triangle* $T_1T_2T_3$, for which $\theta = 0$, an the *circumnormal triangle*

$N_1N_2N_3$, for which $\theta = \pi/2$. These two triangles together form a regular six–pointed star, shown in Figure 3.

Figure 1. Antipedal lines of triangle ABC

Figure 2. Segments representing antipedal lines

Figure 3. Circumtangential triangle $T_1T_2T_3$
and circumnormal triangle $N_1N_2N_3$

The name *antipedal line* is chosen because of the relationship between such a line and a *pedal line*, this latter being a standard term synonymous with *Simson line* and *Wallace line*–the feet of the perpendiculars from X onto sidelines BC, CA, AB are collinear points on the pedal line of X. Now suppose X on Γ has antipedal line XX^{-1} and pedal line L. In Figure 4, let $\angle XAB = \varphi$. Let AY be the reflection of AX about the bisector of $\angle BAC$. By the definition of isogonal conjugate, X^{-1} is on the line AY. Since X^{-1} is at infinity, XX^{-1} is the line through X parallel to AY. It intersects BC at an angle $C - \varphi$, as shown. Now let Q_1, Q_2, Q_2 be the feet of the perpendiculars from X to the respective sides of $\triangle ABC$, so that they are on L. Thus, $\angle XQ_3A$ and $\angle XQ_2A$ are right angles, making XA a diameter of a circle through Q_2 and Q_3. This means that AXQ_3Q_2 is a cyclic quadrilateral, so that $\angle XQ_2Q_3 = \angle XAQ_3 = \varphi$, and $\angle Q_3Q_2C = \pi/2 - \varphi$. Also, $\angle CQ_1Q_2 = \pi - C - (\pi/2 - \varphi)$, which makes this angle complementary to the angle between XX^{-1} and BC. Therefore XX^{-1} is perpendicular to L. Next we define two mappings, Λ and Ψ, in terms of pairs of points P and X. The images, $\Lambda(P, X)$ and $\Psi(P, X)$, always lie on Γ.

Figure 4. Antipedal line of X, perpendicular to pedal line, $Q_1Q_2Q_3$, of X

Definition 1. (The Lambda Mapping.) $\Lambda(P,X)$ is the point $(PX \cap \mathcal{L}^\infty)^{-1}$, for all P and X for which the line PX meets $\mathcal{L}^\infty$ in one point. Note the connection with antipedal lines; if $X \in \Gamma$ then X^{-1} is the point where the antipedal line XX^{-1} meets $\mathcal{L}^\infty$, so that $\Lambda(X, X^{-1}) = X$.

Definition 2. (The Psi Mapping.) Let $Y = \Lambda(P,X)$, and write Y and Z for the points in which the line YP^{-1} meets Γ, where $Z = Y$ if and only if XY is tangent to Γ. Define $\Psi(P,X) = Z$. See Figure 5. Here, too, the connection with antipedal lines is easy to see, for if $X \in \Gamma$ then $Y = \Lambda(X, X^{-1}) = X$, so that $\Psi(X, X^{-1})$ is the point other than X where the antipedal line XX^{-1} meets Γ. We state here some facts about Λ and Ψ, to be proved in greater generality in Section 5.

Figure 5. The mappings Λ and Ψ

Theorem 1. Suppose P and X are points in the extended plane of $\triangle ABC, P \neq X$, and the line of X and $\Psi(P,X)$ meets Γ in two points. Then the two points are $\Lambda(P^{-1}, X^{-1})$ and $\Psi(P,X)$. (See Figure 5.)

Theorem 2. Suppose X is a point on Γ. Let Y be the antipode of X, and let Q be the point of intersection of the antipedal lines of X and Y. Then Q is the antipode of $\Lambda(X,Y)$. (See Figure 6.)

Figure 6. Antipodal pairs X, Y and Q, $\Lambda(X, Y)$

By Theorem 1, if $P \in \Gamma$ and $X \neq P$, then the line PX meets Γ again in the point $\Psi(P, X)$. In particular, if $O =$ circumcenter, then the antipode of P is $\Psi(P, O)$.

3. Trilinear Coordinates. Suppose P is a point in the plane of $\triangle ABC$ having sidelengths a, b, c, and α, β, γ are numbers proportional to the directed distances from P to the sides BC, CA, AB, respectively. Then the *actual directed distances* from P to the sides are $k\alpha, k\beta, k\gamma$, where $k = 2\sigma/(a\alpha + b\beta + c\gamma)$ and $\sigma =$ area of $\triangle ABC$. However, mere ratios $\alpha : \beta : \gamma$ determine P, and any such triple is called *trilinears* for P. For example, the incenter, being equidistant from the sidelines of $\triangle ABC$, has trilinears $1 : 1 : 1$. Standard formulas involving trilinears are given in [1]. In particular, the isogonal conjugate of a point $P = \alpha : \beta : \gamma$, where $\alpha\beta\gamma \neq 0$, is the point $P^{-1} = \alpha^{-1} : \beta^{-1} : \gamma^{-1}$. Also, $A^{-1} = 0 : c : -b, B^{-1} = -c : 0 : a, C^{-1} = b : -a : 0$. The line at infinity, $\mathcal{L}^\infty$, has equation $a\alpha + b\beta + c\gamma = 0$, so that Γ, as the isogonal conjugate of $\mathcal{L}^\infty$, has equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$. A point X on Γ has trilinears given as in [4] by

$$X = \csc\theta : \csc(C - \theta) : -\csc(B + \theta). \tag{1}$$

Here, 2θ is an angle AOX, oriented in the same sense as $\triangle ABC$, as indicated in Figure 7. We shall use this parameterization of Γ in a proof of Theorem 3.

Figure 7. The parameter θ

<u>Theorem 3</u>. The circumtangential triangle, $T_1T_2T_3$, is an equilateral triangle with vertices

$$\begin{aligned} T_1 &= \csc((C - B)/3) : \csc((B + 2C)/3) : -\csc((2B + C)/3), \\ T_2 &= -\csc((2C + B)/3 : \csc((A - C)/3) : \csc((C + 2A)/3), \\ &= \csc((A + 2B)/3) : -\csc((2A + B)/3) : \csc((B - A)/3). \end{aligned} \tag{2}$$

Proof. Equation (2) and the trilinears of X^{-1} yield an equation for the line XX^{-1}:

$$\begin{aligned}&[\csc(B+\theta)\sin(C-\theta)-\csc(C-\theta)\sin(B+\theta)]\alpha\\ &+[\csc\theta\sin(B+\theta)-\csc(B+\theta)\sin\theta]\beta\\ &+[\csc\theta\sin(C-\theta)-\csc(C-\theta)\sin\theta]\gamma=0,\end{aligned}$$

or

$$a\sin\theta\sin(C-B-2\theta)\alpha+b\sin(C-\theta)\sin(B+2\theta)\beta+c\sin(B+\theta)\sin(C-2\theta)\gamma=0.$$

It can be shown that the line tangent to Γ at X is given by the equation

$$(a\sin^2\theta)\alpha+b\sin^2(C-\theta)\beta+c\sin^2(B+\theta)\gamma=0.$$

If this line is XX^{-1}, the trilinears of X^{-1} must satisfy

$$a\sin^3\theta+b\sin^3(C-\theta)+c\sin^3(B+\theta)=0,$$

or

$$a\sin^3\theta+b(\sin C\cos\theta-\cos C\sin\theta)^3+c(\sin B\cos\theta+\cos C\sin\theta)^3=0.$$

This equation reduces to

$$\begin{aligned}&a\sin B\sin C\cos(B-C)\sin^3\theta+3a\sin B\sin C\sin(B-C)\sin^2\theta\cos\theta\\ &-3a\sin B\sin C\cos(B-C)\sin\theta\cos^2\theta-a\sin B\sin C\sin(B-C)\cos^3\theta=0.\end{aligned}$$

Elimination of the common factor and division by $\cos^3\theta$ yield

$$(\tan^3\theta-3\tan\theta)\cos(B-C)+(3\tan^2\theta-1)\sin(B-C)=0,$$

and the identity

$$\tan 3\theta=(3\tan\theta-\tan^3\theta)/(1-3\tan^2\theta)$$

reduces this to

$$\tan 3\theta = -\tan(B-C).$$

This equation has three essentially distinct solutions: $3\theta = -(B-C)$, $\pi-(B-C)$, and $2\pi-(B-C)$, corresponding to $\theta_1 = -(B-C)/3$, $\theta_2 = \pi/3-(B-C)/3$, $\theta_3 = 2\pi/3-(B-C)/3$.

Thus there are three values of θ for which the antipedal line of X is tangent to Γ. They are spaced on the circumcircle Γ at intervals of $\pi/3$ (in terms of θ), or $2\pi/3$ (in terms of 2θ), so that the points of tangency are the vertices of an equilateral triangle.

Let T_1 be the point corresponding to θ_1. Then T_1 and A are on the same arc of BC, since $-B < -(B-C)/3 < C$, where C is the value of the parameter θ for the point B, and $-B$ is that for C. Similarly, T_2 and B are on the same arc of CA, while T_3 and C are on the same arc of AB, therefore justifying the prescribed ordering. Substitution of $-(B-C)/3$ for θ in (2) gives trilinears for T_1 as asserted, and cyclic permutations of variables and subscripts give trilinears for T_2 and T_3.

4. Trilinears and Equilateral Triangles. Probably the best known equilateral triangle associated with an arbitrary triangle is Morley's triangle, described by Morley's Theorem: *The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.* "One of the most astonishing and totally unexpected theorems in mathematics," to quote Oakley and Baker [3], Morley's Theorem obviously involves the angles $A/3, B/3, C/3$, as in Figure 8.

<u>Theorem 4</u>. The circumtangential triangle, $T_1T_2T_3$, is homothetic to Morley's triangle, $M_1M_2M_3$.

<u>Proof</u>. To find trilinears for M_1, note that for any point $\alpha : \beta : \gamma$ on BM_1, we have $x_1/x_3 = \sin(B/3)/\sin(2B/3) = 1/(2\cos(B/3))$. Similarly, if the point is on CM_1, then $x_1/x_2 = 1/(2\cos(C/3))$. Thus, trilinears for M_1 are now determined, and by cyclic permutation of angles and coordinates, we have $M_1 = 1 : 2\cos(C/3) : 2\cos(B/3)$, $M_2 = 2\cos(C/3) : 1 : 2\cos(A/3)$, $M_3 = 2\cos(B/3) : 2\cos(A/3) : 1$. From the coordinates, we obtain an equation for the line M_1M_2.

$$\left(4\cos\frac{C}{3}\cos\frac{A}{3} - 2\cos\frac{B}{3}\right)\alpha + \left(4\cos\frac{B}{3}\cos\frac{C}{3} - 2\cos\frac{A}{3}\right)\beta + \left(1 - 4\cos^2\frac{C}{3}\right)\gamma = 0.$$

Figure 8. The circumtangential triangle
is homothetic to Morley's triangle

In order to prove that M_1M_2 is parallel to T_1T_2, it suffices to prove that M_1M_2 is parallel to the antipedal line of T_3. But, since T_3^{-1} is the point at infinity on $T_3T_3^{-1}$, it suffices to prove that T_3^{-1} lies on M_1M_2. To prove this, we substitute the coordinates of T_3^{-1} into the equation of M_1M_2:

$$-\left(4\cos\frac{C}{3}\cos\frac{A}{3}-2\cos\frac{B}{3}\right)\sin\frac{A+2B}{3}$$

$$+\left(4\cos\frac{B}{3}\cos\frac{C}{3}-2\cos\frac{A}{3}\right)\sin\frac{2A+B}{3}$$

$$+\left(1-4\cos^2\frac{c}{3}\right)\sin\frac{A-B}{3},$$

which equals zero. This completes a proof that $M_1M_2 \parallel T_1T_2$. Symmetry shows that $M_2M_3 \parallel T_2T_3$ and $M_3M_1 \parallel T_3T_1$. (See Figure 8.) Thus, the triangles $M_1M_2M_3$ and $T_1T_2T_3$ are homothetic.

Let $z = \sin(A/3)\sin(B/3)\sin(C/3)$. The sidelength of Morley's triangle is $8Rz$, where R is the circumradius of $\triangle ABC$. It follows that the ratio of homothety of the circumtangential triangle to Morley's triangle is $8z/\sqrt{3}$.

The circumnormal triangle, $N_1N_2N_3$, can be described as the triangle whose vertices are the points X on Γ for which the antipedal line XX^{-1} is normal to Γ, or, equivalently, those X for which XX^{-1} passes through the circumcenter, O. An equation for the line OX found via (2) and (1) is $[\csc(C-\theta)\cos C+\csc(B+\theta)\cos B]\alpha-[\csc(B+\theta)\cos A+\csc\theta\cos C]\beta+[\csc\theta\cos B-\csc(C-\theta)\cos A]\gamma=0$. In order for this to be the antipedal line of X, it must contain the point X^{-1}. When trilinears for X^{-1} are substituted into the equation, the procedure used in proving Theorem 1 yields roots $\theta_1' = \pi/2-(B-C)/3, \theta_2' = 5\pi/6-(B-C)/3, \theta_3' = \pi/6-(B-C)/3$. These values in (2) give trilinears for the vertices $N_1N_2N_3$, respectively, and they establish the fact that N_i is the antipode of T_i for $i=1,2,3$

so that $N_1N_2N_3$ is also an equilateral triangle. It, too, of course, is homothetic to Morley's triangle.

5. Trilinears and Antipedal Lines. The envelope of the system of antipedal lines is of some significance. To determine this curve, first consider Figure 9, in which BX is a cevian of $\triangle ABC$, making an angle φ with BC. Let h be the length of BX, so that the distances from X to BC and AB are $h\sin\varphi$ and $h\sin(B-\varphi)$, respectively. Therefore an equation for BX is $\sin(B-\varphi)\alpha-(\sin\varphi)\gamma=0$. This line meets $\mathcal{L}^\infty$ in the point

$$\sin\varphi : -\sin(C+\varphi) : \sin(B-\varphi). \tag{3}$$

Therefore, (3) is the point at infinity on any line making the angle φ with BC, in the sense shown in Figure 9.

Figure 9. The angle θ associated with cevian BX

Since X^{-1} is the point at infinity on the antipedal line of X, its trilinears must be proportional to those of (3). Hence we have $\sin\varphi/\sin\theta = -\sin(C+\varphi)/\sin(C-\theta) = -\sin(B-\varphi)/\sin(B+\theta)$, for which the solution is $\varphi = -\theta$ or $\varphi = \pi - \theta$. Therefore the inclination of XX^{-1} with respect to BC is $-\theta$, which means that as X moves around Γ with angular velocity $2d\theta/dt$, its antipedal line turns with angular velocity $-d\theta/dt$. The result as in [5] is that the envelope of the antipedal lines is a deltoid, or hypocycloid of three cusps, circumscribed about Γ.

This hypocycloid resembles Steiner's deltoid, which is the envelope of the pedal lines and is circumscribed about the nine–point circle. In fact, it is easy to prove that Γ is the nine–point circle of the excentral triangle, whose vertices are the excenters ($I_1 = -1:1:1, I_2 = 1:-1:1, I_3 = 1:1:-1$) of $\triangle ABC$. Finally, it can be shown that the antipedal lines of $\triangle ABC$ are the pedal lines of $I_1I_2I_3$.

6. Trilinears and the Two Mappings. We begin by formulating the mappings Λ and Ψ introduced in Section 1. Let $P = u:v:w$ and $X = x:y:z$. Form $D = vz - wy, E = wx - uz, F = uy - vx$, and define

$$\text{(4)}\quad \Lambda(P,X) = (aE-bD)(aF-cD) : (bF-cE)(bD-aE) : (cD-aF)(cE-bF)$$

$$\text{(5)}\qquad \Psi(P,X) = avwEF : bwuFD : cuvDE.$$

Formula (4) comes directly from the equation $D\alpha + E\beta + F\gamma = 0$ for line PX and that the fact that this line meets $\mathcal{L}^\infty$ in the point whose first coordinate is $bF - cE$. Formula (5) is a bit more troublesome. The point on the right–hand side clearly lies on the circumcircle $a/\alpha + b/\beta + c/\gamma = 0$ and is also collinear with P^{-1} and $\Lambda(P,X)$, as attested by (1), since

$$\text{(6)}\qquad \begin{vmatrix} 1/u & 1/v & 1/w \\ a/uD & b/vE & c/wF \\ 1/(bF-cE) & 1/(cD-aF) & 1/(aE-bD) \end{vmatrix} = 0.$$

It appears that the most practical way to evaluate the determinant is by a computer algebra system, treating a,b,c,u,v,x,y,z as indeterminates. We prefer (6) to a purely geometric argument because (6) includes the geometric result as a special case of a larger truth; that is, a, b, c need not be sidelengths of a triangle — indeed, they and the other six numbers can be any complex numbers, with certain obvious exceptions.

We now give a few details about each mapping. To form $\Lambda(P,X)$, we must have $X \neq P$, and P and X cannot both lie on $\mathcal{L}^\infty$ unless they are identical. The set of all Y such that $\Lambda(P,Y) = \Lambda(P,X)$ is simply the line PX with the point P deleted. Thus, if $\mathcal{C}$ is any simple closed curve such that P lies in the interior of $\mathcal{C}$, then as X moves around $\mathcal{C}$ once, the point $\Lambda(P,X)$ moves around Γ once. To form $\Psi(P,X)$, we must again have $X \neq P$, and P and X both must not lie on either $\mathcal{L}^\infty$ or Γ. The set of all Y satisfying $\Psi(P,Y) = \Psi(P,X)$ is again the line PX with P deleted.

Theorem 1. (Generalized.) Suppose $a,\ b,\ c;\ u,\ v,\ w;\ x,\ y,\ z$ are indeterminates over the complex number field, and $D = vz - wy, E = wx - uz, F = uy - vx$. Then

$$\begin{vmatrix} x & y & z \\ a/uD & b/vE & c/wF \\ u/(bwzF - cvyE) & v/(cuxD - awzF) & w/(avyE - buxD) \end{vmatrix} = 0.$$

Proof. That the determinant vanishes can be easily established using a computer algebra system. We shall now see why this vanishing proves Theorem 1 as stated in Section 1. Suppose $a,\ b,\ c$ are sidelengths of a $\triangle ABC$, $P = u : v : w$, and $X = x : y : z$. Then row 1 consists of trilinears for X, row 2, those of $\Psi(P,X)$, and row 3, those of $\Lambda(P^{-1}, X^{-1})$. By (1), the points given by these rows are collinear. Since $\Psi(P,X)$ and $\Lambda(P^{-1}, X^{-1})$ lie on the circumcircle, Γ, of $\triangle ABC$, they must be the two points in which the line of X and $\Psi(P,X)$ meets Γ.

Theorem 2. (Generalized.) Suppose a, b, c, x', y' are indeterminates over the complex number field. Let

(7) $$q = ay' + bx',\ x = qx',\ y = qy',\ z = -cx'y'$$

(8) $$a' = a'(a,b,c) = (b^2 + c^2 - a^2)/(2bc), b' = a'(b,c,a), c' = a'(c,a,b)$$

$$x_1 = \frac{a}{x(yb' - zc')}, \quad y_1 = \frac{b}{y(zc' - xa')}, \quad z_1 = \frac{c}{z(xa' - yb')};$$

$$l_1 = x(y^2 - z^2), \quad m_1 = y(z^2 - x^2), \quad n_1 = z(x^2 - y^2);$$
$$l_2 = x_1(y_1^2 - z_1^2), \quad m_2 = y_1(z_1^2 - x_1^2), \quad n_2 = z_1(x_1^2 - y_1^2);$$
$$x_2 = m_1 n_2 - n_1 m_2, \quad y_2 = n_1 l_2 - l_1 n_2, \quad z_2 = l_1 m_2 - m_1 l_2;$$
$$l_3 = yz_1 - zy_1, \quad m_3 = zx_1 - xz_1, \quad n_3 = xy_1 - yx_1,$$

$$x_3 = \frac{1}{bn_3 - cm_3}, \quad y_3 = \frac{1}{cl_3 - an_3}, \quad \text{and} \quad z_3 = \frac{1}{am_3 - bl_3}.$$

Then

$$\begin{vmatrix} a' & b' & c' \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Proof. Again, vanishing of the determinant is established by a computer algebra system. We shall now see why this vanishing proves Theorem 2 as stated in Section 1. Suppose a, b, c are sidelengths of a $\triangle ABC$. Then by the Law of Cosines, a', b', c' are $\cos A, \cos B, \cos C$, so that the circumcenter, O of $\triangle ABC$ has trilinears $a' : b' : c'$. The circumcircle, Γ, has equation $a/\alpha + b/\beta + c/\gamma = 0$, and it is easy to see that a point $X = x : y : z$ lies in $\Gamma - \{A, B, C\}$ if and only if the coordinates are given as in (7) for some nonzero real numbers x' and y'.

The point $Y = x_1 : y_1 : z_1$ in (8) is found by Corollary 2 to be the antipode of X. The lines $l_i\alpha + m_i\beta + n_i\gamma = 0$, for $i = 1$ and 2, are the antipedal lines X and Y, and they meet in the point $W = x_2 : y_2 : z_2$. Meanwhile, line XY has equation $l_3\alpha + m_3\beta + n_3\gamma = 0$ and meets $\mathcal{L}^\infty$ in the point $Z = x_3 : y_3 : z_3$. The determinant equation therefore implies that the three points, O, W, Z are collinear. Since, as is easily checked, W and Z lie on Γ, the geometric special case of Theorem 2 as stated in Section 1 is now proved.

In the following list of formulations, we use notations introduced in [1]: X_9, X_{10} are, respectively, the Mittenpunkt and the Spieker center. The point X_{100} is given by $\alpha : \beta : \gamma = 1/(b-c) : 1/(c-a) : 1/(a-b)$, which is abbreviated as Γ_{13} below as simply $\alpha = 1/(b-c)$, where β and γ can be figured out from α by the cyclic substitutions $a \to b, b \to c, c \to a$. With this understanding about α as a function of a, b, c that determines β and γ, we list fourteen special points on Γ in abbreviated form.

$$\Gamma_1 = \Lambda(I, H). \quad \alpha = \frac{1}{b(\sec B - \sec A) - c(\sec A - \sec C)}.$$

$$\Gamma_2 = \Lambda(G, O) = X_{74}. \quad \alpha = \frac{1}{\cos A - 2\cos B \cos C}.$$

(Note: GO is the Euler line of $\triangle ABC$.)

$$\Gamma_3 = \Psi(X_{101}, O). \quad \alpha = \frac{a}{(a-b)(a^2+b^2-c^2) - (c-a)(c^2+a^2-b^2)}.$$

$$\Gamma_4 = \Lambda(I, O) = \Psi(X_{101}, X_9). \quad \alpha = \frac{1}{-1 + \cos B + \cos C}.$$

$$\Gamma_5 = \Lambda(O, K) = \text{Tarry point} = X_{98} = \Psi(X_{101}, X_{10}). \quad \alpha = \frac{bc}{b^4 + c^4 - a^2b^2 - a^2c^2}.$$

$$\Gamma_6 = \Lambda(I,K) = \Psi(X_{101}, I). \quad \alpha = \frac{1}{ab + ac - b^2 - c^2}.$$

$$\Gamma_7 = \Lambda(I,G) = \Psi(X_{101}, K). \quad \alpha = \frac{1}{2a - b - c}.$$

$$\Gamma_8 = \Psi(H,K). \quad \alpha = \frac{1}{\cos A(\sin 2B - \sin 2C)}.$$

$$\Gamma_9 = \Psi(O,I) = \Psi(I,H). \quad \alpha = \frac{a}{\sec B - \sec C}.$$

$$\Gamma_{10} = \Psi(I,O). \quad \alpha = \frac{a}{\cos C - \cos B}; \text{ also } \alpha = \frac{a}{(b-c)(b+c-a)}.$$

$$\Gamma_{11} = \Psi(K,O). \quad \alpha = \frac{1}{b\cos C - c\cos B}.$$

$$\Gamma_{12} = X_{101} = \Psi(I,K). \quad \alpha = \frac{a}{b-c}.$$

$$\Gamma_{13} = X_{100} = \Psi(K,I). \quad \alpha = \frac{1}{b-c}.$$

$$\Gamma_{14} = \text{Steiner point} = X_{99} = \Psi(K,G). \quad \alpha = \frac{1}{a(b^2-c^2)}.$$

Several collinearities involving I, G, O, H, K are shown in Graph 1.

Graph 1.

Graph 1 signifies a journey along line segments that cross the circumcircle. Start at Γ_1, go to the circumcenter, O, and continue until meeting Γ_{10}; then go through the incenter, I, over to Γ_4, and so on. This journey visits nine of the special points Γ_i. The other five points can be visited as shown in Graph 2. Note that Graphs 1 and 2 indicate that Γ_{i+9} is the antipode of Γ_i for $i = 1, 2, 3, 4, 5$.

Graph 2.

Figure 10 is another representation of the collinearities indicated by Graph 2.

Figure 10. Collinearities as in Graph 2

7. Ordering of Special Points on the Circumcircle $(A < B < C < \pi/2)$. The main objective here is to show that the fourteen notable points Γ_i are positioned around the circumcircle so that the order in which they occur stays the same for all triangles ABC satisfying $A < B < C < \pi/2$. One way to describe the ordering is by the angle AOX, oriented in the same sense as $\triangle ABC$. Accordingly, we write $X < Y$ if and only if $\angle AOX < \angle AOY$.

However, these angles are not conveniently expressed in trilinears, so we rely on an equivalent way to keep track of points on Γ. For points X and Y on Γ and on the same side of AA' as the point B, as in Figure 11, it is clear that $X < Y$ if and only if $|XT_A| < |YT_A|$, where $|XT_A|$ denotes the distance from X to the line tangent to Γ at A. For $X = x : y : z$ we have $|XT_A| = (bz + cy)k/a$, where $k = 2\sigma/(ax + by + cz)$, so that $x_1 : y_1 : z_1 = X_1 < X_2 = x_2 : y_2 : z_2$ if and only if

Figure 11. Angles AOX and AOY match perpendicular distances $|XT_A|$ and $|YT_A|$

$$
\frac{bz_1 + cy_1}{ax_1 + by_1 + cz_1} < \frac{bz_2 + cy_2}{ax_2 + by_2 + cz_2}. \tag{9}
$$

Presumably, (9) can be used as a basis for proving numerous geometric inequalities missing from [2]. It appears likely that some of these proofs would be lengthy. For now, we report only the results of computer sampling of (9) using many thoughtfully chosen triangles that satisfy $A < B < C < \pi/2$. (We would like to see proofs!)

Graph 3 shows the ordering of twelve points: the vertices A, B, C, their antipodes A', B', C', and the vertices of the circumtangential and circumnormal triangle $T_1T_2T_3$ and $N_1N_2N_3$.

Graph 3.
The vertices are points on Γ. For each pair X and Y of vertices, $X < Y$ if and only if X is to the left of Y.
(See Figure 3.)

Graph 4 shows the ordering of twenty points: the vertices A, B, C, their antipodes A', B', C', and the fourteen notable points Γ_i.

Graph 4.
The vertices are points on Γ, and $X < Y$ if and only if it is possible to follow a path from X to Y without moving on an edge toward the left side of the page.

Graph 5 shows the ordering of twenty points: the vertices of the circumtangential and circumnormal triangles, $T_1T_2T_3$ and $N_1N_2N_3$, and the fourteen notable points Γ_i.

Graph 5.
The vertices are points on Γ, and $X < Y$ if and only if it is possible to follow a path from X to Y without moving on an edge toward the left side of the page.

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