# A POLYNOMIAL CURVE OF CONSTANT WIDTH

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**1. Introduction.** Students are usually amazed to learn that there are closed curves with constant width other than a circle. (The *width* of a closed convex curve in the direction $\theta$ is the distance between the two support lines that are perpendicular to the line making an angle $\theta$ with the positive $x$-axis. See Figure 1. A closed convex curve is said to have *constant width* if its width is the same in all directions.)

Figure 1.

The first such curves that a student sees are the Reuleaux Triangle and the Reuleaux Polygons [2]. These curves, however, are not smooth. These can be modified to produce related curves of constant width that do have a smooth boundary [1], [5]. However, the equations for these curves are not easy to write down.

It is the purpose of this note to exhibit a polynomial equation whose graph is a (noncircular) curve of constant width. We will also find parametric equations for the curve, thus making it easy for students to graph the curve with, say, a graphing calculator.

**2. The Support Function.** To exhibit the desired curve of constant width, we will first need to know a little about the support function for a closed convex curve [1].

Suppose $C$ is a smooth closed convex curve that contains the origin, $O$, in its interior. Consider an arbitrary tangent to the curve and drop a perpendicular $OH$ to this tangent from the origin. (See figure 2.) If the ray $\overrightarrow{OH}$ makes an angle $\theta$ with the positive $x$-axis, then let $p(\theta) = OH$, the signed distance from the origin to the tangent along the direction $\theta$. The function $p(\theta)$ is called the support function for the curve [4]. Given the support function of a curve, let us see how we can find the equation for that curve.

Figure 2.

Since the curve is smooth and convex, it will be the envelope of all its tangents. Let us recall how to find the envelope of a set of lines. The following result is well-known [6].

Lemma. If $f(x, y, \alpha) = 0$ is a family of straight lines, then the equation for the envelope of these lines can be obtained by eliminating the variable $\alpha$ from the two equations $f(x, y, \alpha) = 0$ and $\frac{d}{d\alpha} f(x, y, \alpha) = 0$.

In Figure 2, let the coordinates of point $P$ be $(x, y)$. (This is the point where the tangent touches the curve.) The coordinates of point $H$ is then seen to be $(p(\theta)\cos\theta, p(\theta)\sin\theta)$. The equation of the tangent is easily found to be

$$\frac{y - p(\theta)\sin\theta}{x - p(\theta)\cos\theta} = -\cot\theta. \tag{1}$$

Applying the lemma to this family of tangents (with $\alpha = \theta$), we differentiate with respect to $\theta$. From that result and equation (1), we can solve for $x$ and $y$ in terms of $\theta$. We thereby obtain the following result [4].

Proposition. If $p(\theta)$ is the support function for a closed convex curve $C$, then the parametric equations for $C$ are given by

$$\begin{aligned} x &= p(\theta)\cos\theta - p'(\theta)\sin\theta \\ y &= p(\theta)\sin\theta + p'(\theta)\cos\theta. \end{aligned}$$

**3. Curves of Constant Width.** The width of a closed convex curve in the direction $\theta$ is easily seen to be $w(\theta) = p(\theta) + p(\theta + \pi)$. If we could therefore find a simple function $p(\theta)$ such that $p(\theta) + p(\theta + \pi)$ is a constant independent of $\theta$, then we would obtain a curve of constant width. But this is true for the function $p(\theta) = a\cos^2(k\theta/2) + b$ where $k$ is any odd positive integer and $a, b \geq 0$, since $\cos(\phi + k\pi/2) = \pm\sin\phi$ and $\sin^2\phi + \cos^2\phi = 1$. This idea was suggested by [3].

If we let $p(\theta) = a\cos^2(\theta/2) + b$, then the curve turns out to be a circle, the simplest of all curves of constant width.

If we let $p(\theta) = a\cos^2(3\theta/2) + b$, then the curve obtained is not always convex. Choosing $a = 2$ and $b = 8$ gives a pleasing convex curve as shown in Figure 3.

Figure 3.

In this case, we find that the parametric equations are equivalent to

$$\begin{aligned} x &= 9\cos\theta + 2\cos 2\theta - \cos 4\theta \\ y &= 9\sin\theta - 2\sin 2\theta - \sin 4\theta \end{aligned} \tag{2}$$

and the curve is traced as $\theta$ varies from 0 to $2\pi$.

To get an equation between $x$ and $y$, we can proceed as follows. Expanding everything in terms of $c = \cos\theta$ and $s = \sin\theta$, we get

$$\begin{aligned} x &= -3 + 9c + 12c^2 - 8c^4 \\ y &= s(9 - 4c - 4c^3 + 4cs^2). \end{aligned}$$

Squaring the second equation and substituting in $s^2 = 1 - c^2$ gives us two equations in the three unknowns $x$, $y$, and $c$. We can therefore eliminate $c$; but this is very messy. Using Mathematica, we can find that resultant. We state the result as the following theorem.

Theorem. The graph of the polynomial equation

$$\begin{aligned} &(x^2+y^2)^4 - 45(x^2+y^2)^3 - 41283(x^2+y^2)^2 + 7950960(x^2+y^2) \\ &+ 16(x^2-3y^2)^3 + 48(x^2+y^2)(x^2-3y^2)^2 \\ &+ (x^2-3y^2)x[16(x^2+y^2)^2 - 5544(x^2+y^2) + 266382] = 720^3 \end{aligned}$$

is a closed convex curve of constant width (see Figure 3).

**4. Open Questions.** The polynomial curve found is pretty complicated. Can it be put in simpler form? Perhaps if we choose as the support function, a linear combination of functions of the form $\cos^2(k\theta/2)$ for odd $k$, a simpler result can be obtained. Our polynomial is of degree 8. Is there one with lower degree? What is the lowest degree polynomial whose graph is a noncircular curve of constant width?

### *References*

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