PRODUCTS AND SUMS WITH APPLICATIONS

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Abstract. The Twin Prime Conjecture states that the number of twin primes is infinite. Many attempts to prove or disprove the conjecture have failed. The objective of this note is to tie the Twin Prime Conjecture to complex variable theory and prove some results that make it possible to consider the conjecture from a complex variable viewpoint rather than from a purely number theoretic one.

1. Introduction. Prime numbers differing by 2 are called twin primes. Whether twin primes are finite or infinite in number is one of the most famous problems in number theory. Brun [1] showed that

$$\sum_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \frac{1}{p}$$

is finite. In fact, the sum

$$\sum_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \left(\frac{1}{p} + \frac{1}{p+2}\right),$$

known as Brun's Constant B, has been calculated by Shanks and Wrench [4] and by Brent [5] to be approximately 1.90216054. This paper originated from an idea where the author related the Twin Prime Conjecture to complex variable theory in the following way. Since $B < \infty$, the canonical product

$$\prod_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \left(1 - \frac{z}{p}\right)$$

is an entire function [3]. Moreover, if the number of twin primes was finite, then

$$\prod_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \left(1 - \frac{z}{p}\right)$$

would be a real polynomial and thus of order 0 [3]. Consequently, if

$$\prod_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \left(1 - \frac{z}{p}\right)$$

has a nonzero order, then the number of twin primes is infinite. We shall say something about the order of

$$\prod_{\substack{p \text{ and } p+2 \\ \text{are prime}}} \left(1 - \frac{z}{p}\right)$$

later in this paper.

2. Results. Suppose $\{p_n\}$ is a sequence of positive numbers (the sequence may be finite). For infinite sequences we suppose further that $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges. (If $\{p_n\}_{n=1}^{k}$ is a finite sequence, then the corresponding convergence condition that $\sum_{n=1}^{k} \frac{1}{p_n}$ converges is clear.) Finally, let $A = \sum_n \frac{1}{p_n}$. The canonical product

$$\prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n}$$

is an entire function [3]. We can now state the following theorem.

Theorem 1. Let $\{p_n\}$ be a sequence such that $\sum_n \frac{1}{p_n}$ converges. Then the order of

$$\prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n}$$

is 1.

Proof. The canonical product

$$\prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n}$$

is an entire function [3]. Let $a > 1$. Since $p_n^a > p_n$, $\sum_n \frac{1}{p_n^a} < \infty$. Thus, $\sum_n \frac{1}{p_n^a} < \infty$, when $a \geq 1$. Consequently, the genus of the zeros of $\prod_n (1 - \frac{z}{p_n})$ is 0. By the Hadamard Factorization Theorem and Boas [2], the genus of $\prod_n (1 - \frac{z}{p_n})$ is also 0. From Young [3], $\prod_n (1 - \frac{z}{p_n})$ is of exponential type 0. In particular, since $\prod_n (1 - \frac{z}{p_n})$ is of exponential type, it is of order $k \leq 1$.

We now consider two cases.

Case 1. $k < 1$. Since the order of e^{Az} is 1, it follows from Levin [6] that

$$\prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n} = e^{Az} \prod_n \left(1 - \frac{z}{p_n}\right)$$

is 1.

Case 2. $k = 1$. Since the definition of type and exponential type agree for functions of order 1, the type of $\prod_n (1 - \frac{z}{p_n})$ is 0. Clearly, the type of e^{Az} is $A > 0$. Thus, from Levin [6], the order of

$$\prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n} = e^{Az} \prod_n \left(1 - \frac{z}{p_n}\right)$$

is 1. The proof of Theorem 1 is complete.

We now extend our functions to the class of meromorphic functions (for the definition of the order of a meromorphic function and its consistency with that of the order of an entire function see, for instance, [7]).

The following result is a Corollary of Theorem 1.

Corollary. Let $\{p_n\}$ be a sequence such that $\sum_n \frac{1}{p_n}$ is finite. The order of $\sum_n \frac{1}{z - p_n}$ is less than or equal to 1.

Proof. Let

$$f(z) = \prod_n \left(1 - \frac{z}{p_n}\right) e^{z/p_n}.$$

By Theorem 1, $f(z)$ is of order 1. Moreover

$$\log f(z) = Az + \sum_n \log\left(1 - \frac{z}{p_n}\right).$$

Differentiating with respect to z we get

$$f'(z) = f(z)\left(A + \sum_n \frac{1}{z - p_n}\right).$$

To get a contradiction, suppose that

$$\sum_n \frac{1}{z - p_n}$$

is of order greater than 1. Then by [6],

$$f(z) \sum_n \frac{1}{z - p_n}$$

is of order greater than 1. So $f'(z)$ is of order greater than 1. But a function and its derivative should have the same order. This is a contradiction and the proof of the corollary is complete.

3. Concluding Remarks.

I. If we now consider the particular sequence $\{p_n\}$ of twin primes, we see from the proof of Theorem 1 that $\prod_n (1 - \frac{z}{p_n})$ has genus 0 which, in turn, implies that $\prod_n (1 - \frac{z}{p_n})$ is of order less than or equal to 1.

Can one use the fact that $\prod_n (1 - \frac{z}{p_n})$ is of genus 0 more effectively to obtain a sharper result on the order of $\prod_n (1 - \frac{z}{p_n})$? More precisely, if one could show that the order of $\prod_n (1 - \frac{z}{p_n})$ is nonzero, that would imply that there are infinitely many twin primes.

II. The sequence $\{q_n\}$ of perfect numbers is known to satisfy $\sum_{n=1}^{\infty} \frac{1}{q_n} < \infty$, a condition similar to Brun's. Since Theorem 1 and its corollary hold for any sequence $\{p_n\}$ of positive numbers for which $\sum_n \frac{1}{p_n} < \infty$, we have the order of

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{q_n}\right) e^{z/q_n}$$

is 1 and the order of $\sum_{n=1}^{\infty} \frac{1}{z - q_n}$ is less than or equal to 1.

References

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