A RELATIONSHIP BETWEEN CHORDS AND THE RADIUS OF A CIRCLE

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Let a, b, and c represent the distinct lengths of the chords subtending arcs AB, BC, and CD, respectively. Assume that arc AD forms a semicircle (see Figure 1).

Figure 1.

The purpose of this paper is to determine the length of the radius of the semicircle as a function of a, b, and c. The level of the paper is for undergraduate students.

Let $r \neq 0$ represent the length of the radius of the semicircle. Ptolemy's Theorem [2] asserts that in a cyclic quadrilateral, the product of the lengths of the diagonals is equal to the sum of the products of the lengths of the two pairs of opposite sides. Hence, $(AC)(BD) = (AB)(CD) + (AD)(BC)$ and so

$$(AC)(BD) = ac + 2rb. \tag{1}$$

Since triangles ACD and ABD are right triangles,

$$AC = \sqrt{4r^2 - c^2} \quad \text{and} \quad BD = \sqrt{4r^2 - a^2}.$$

So, identity (1) becomes

$$\sqrt{(4r^2 - c^2)(4r^2 - a^2)} = ac + 2rb.$$

Squaring both sides of this equation and simplifying gives

$$r^3 - (a^2 + b^2 + c^2)r/4 - abc/4 = 0. \tag{2}$$

By Descartes rule of sign, equation (2) has one positive root and that is the solution that is desired. Before determining this root, a brief review of Cardano's formula [1] will be given.

The roots of a cubic equation of the form

$$x^3 + 3px + q = 0 \tag{3}$$

depend upon the sign of $4p^3 + q^2$. If $4p^3 + q^2 \geq 0$, the solution set of (3) is

$$\{\sqrt[3]{m} + \sqrt[3]{n},\ \omega\sqrt[3]{m} + \omega^2\sqrt[3]{n},\ \omega^2\sqrt[3]{m} + \omega\sqrt[3]{n}\}$$

where

$$\omega = (-1 + i\sqrt{3})/2, \quad m = (-q + \sqrt{4p^3 + q^2})/2,$$

and

$$n = (-q - \sqrt{4p^3 + q^2})/2.$$

If $4p^3 + q^2 < 0$, then write m in the polar form $m = |m|\ e^{i\theta}$ (and $n = \overline{m}$). The solution set of (3) is

$$\{2\sqrt[3]{|m|}\cos(\theta/3),\ 2\sqrt[3]{|m|}\cos[(\theta + 2\pi)/3],\ 2\sqrt[3]{|m|}\cos[(\theta + 4\pi)/3]\}.$$

To relate these results to (2), let $p = -(a^2 + b^2 + c^2)/12$ and $q = -abc/4$. By the arithmetic mean–geometric mean inequality,

$$(a^2 + b^2 + c^2)/3 > \sqrt[3]{a^2b^2c^2},$$

which is equivalent to

$$(a^2 + b^2 + c^2)^3 > 27a^2b^2c^2.$$

That is, $4p^3 + q^2 < 0$. Strict inequality was used above since a, b, and c are not equal.

Now,

$$m = [3abc + i\sqrt{[(a^2 + b^2 + c^2)^3 - 27a^2b^2c^2]/3}]/24$$

and so

$$|m| = \frac{\sqrt{(a^2 + b^2 + c^2)^3/3}}{24}$$

and

$$\theta = \operatorname{Arctan}\left[\frac{\sqrt{[(a^2 + b^2 + c^2)^3 - 27a^2b^2c^2]/3}}{3abc}\right].$$

Therefore, the positive root of (2) is

$$(4) \qquad r = \sqrt[3]{\frac{1}{3}\sqrt{(a^2 + b^2 + c^2)^3/3}}$$

$$\cdot \cos\left\{\frac{1}{3}\operatorname{Arctan}\left[\frac{\sqrt{[(a^2 + b^2 + c^2)^3 - 27a^2b^2c^2]/3}}{3abc}\right]\right\}.$$

If the restriction of distinctness is removed and one lets $a = b = c$, then equation (2) becomes $r^3 - 3a^2r/4 - a^3/4 = 0$ and so $(r - a)(r + a/2)^2 = 0$. The positive root is $r = a$ which agrees with (4) when $a = b = c$. Geometrically, this means that to inscribe a regular hexagon in a circle, use the radius of the circle to mark the vertices on the circle.

References

1. *CRC Standard Mathematical Tables*, 21st ed., 103–104.
2. D. C. Kay, *College Geometry – A Discovery Approach*, Harper Collins, (1994), 31.

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