SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

93. [1996, 89] *Proposed by Alan H. Rapoport, Ashford Medical Center, Santurce, Puerto Rico.*

(a) What is the minimum number of people needed in order to assure a greater than 50% chance of at least three having the same birthday?

(b) What is the minimum number of people you must query in order to have a 50% chance of finding at least two having your birthday?

Solution by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

(a) We can model this situation as an ordered random sample of size n with replacement from a finite population of N distinct objects. There are N^n equally likely outcomes in the sample space.

Let B denote the event that at least one object is selected three or more times in the sample. Then $\overline{B}$ is the event that any item is selected at most twice. We will find an expression for $P(\overline{B})$.

Let A_0 denote the event that all n objects selected in the sample are distinct. Then,

$$P(A_0) = \binom{N}{n} n!/N^n.$$

Let A_1 be the event that exactly one object is selected twice and the other $n-2$ objects selected are distinct. Then there are $\binom{N}{n-1}$ ways to select the $n-1$ objects, $\binom{n-1}{1}$ ways to pick the "double", and $n!/2$ ways to form an ordered arrangement. So,

$$P(A_1) = \binom{N}{n-1}\binom{n-1}{1} n!/(2N^n).$$

In general for j, $2 \leq j \leq n/2$, let A_j denote the event that exactly j objects are selected twice and the other $n-2j$ objects are distinct. Then there are $\binom{N}{n-j}$

ways to choose the $n-j$ items for the sample, $\binom{n-j}{j}$ ways to select the j "doubles", and $n!/2^j$ ways to form an ordered arrangement. So,

$$P(A_j) = \binom{N}{n-j}\binom{n-j}{j} n!/(2^j N^n).$$

Since $A_0, A_1, \ldots, A_m$ (where m is the greatest integer in $n/2$) partitions $\overline{B}$, we have

$$P(\overline{B}) = (n!/N^n)\sum_{j=0}^{m}\binom{N}{n-j}\binom{n-j}{j}/2^j.$$

With $N = 365$, a computer check yields

$$P(\overline{B}) > \frac{1}{2} \text{ for } n \le 87 \text{ and } P(\overline{B}) < \frac{1}{2} \text{ for } n \ge 88.$$

So 88 is the minimum number of people needed to assure a greater than 50% chance of at least three having the same birthday.

This result along with an equivalent expression for $P(\overline{B})$, attributed to C. F. Pinzka, is reported on page 830 of the October 1960 issue of the *American Mathematical Monthly.* Donald E. Knuth gives essentially the same expression for $P(\overline{B})$, on page 195 of the August-September 1977 issue of *Crux Mathematicorum.*

(b) We can model this problem as a sequence of independent Bernoulli trials with probability p of matching your birthday on each trial. The complement of at least two having your birthday is at most one of the n people matches your birthday.

Let $\overline{B}_n$ denote the complement. Then

$$\begin{aligned} P(\overline{B}_n) &= q^n + npq^{n-1} \\ &= q^{n-1}(q+np) = q^{n-1}(1+(n-1)p), \end{aligned}$$

where $q = 1-p$.

We can define $P(\overline{B}_n)$ recursively as follows.

$$P(\overline{B}_3) = q^2(1+2p)$$

$$P(\overline{B}_{n+1}) = q(1+np)P(\overline{B}_n)/(q+np), \quad \text{for } n \geq 3.$$

For a given p, compute $P(\overline{B}_n)$ recursively until

$$P(\overline{B}_n) \leq \frac{1}{2}.$$

Listed below is a brief program for the TI 80, TI 82, or TI 83.

Program	Comment
:Prompt P	input the probability p
:$1 - P \to Q$	compute and store q
:$Q^2(2P+1) \to S$	compute $P(\overline{B}_3)$ and store in S
:$3 \to N$	initialize counter
:While $(S > 0.5)$	continue until $P(\overline{B}_n) \leq 0.5$
:$Q(1 + N * P)S/(Q + N * P) \to S$	recursive formula
:$N + 1 \to N$	increment counter
:End	end of the while loop
:Disp N	print out N and S.
:Disp S	

With $p = 1/365$, the output is $n = 613$ and $P(\overline{B}_n) = 0.49936$.

For a February 29 leap year birthday, set $p = 1/1461$, the output is $n = 2452$ and $P(\overline{B}_n) = 0.49994$.

Comment by the proposer. The following is a list of references for the birthday problem.

References

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10. R. Pinkham, "The Birthday Problem; Pairs and Triples," *The Mathematical Gazette*, 69, (1985), 279.
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13. Solution to Problem 70.B, *The Mathematical Gazette*, 70 (1986), 228–229.
14. Solution to Problem 195, *Crux Mathematicorum*, 3 (1977), 83–87.
15. Solution to Problem 195, *Crux Mathematicorum*, 3 (1977), 195–196.

Also solved by the proposer.

94. [1996, 90] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let n be a nonnegative integer. Prove

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = -F_n,$$

where F_n denotes the nth Fibonacci number.

Solution I by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Thomas C. Leong, The City College of City University of New York, New York, New York; James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; Lawrence Somer, The Catholic University of America, Washington, D.C.; and Kenneth Davenport, Pittsburgh, Pennsylvania.

Our solution will use the Binet form for F_j:

$$F_j = \frac{\alpha^j - \beta^j}{\sqrt{5}}$$

for each nonnegative integer j, where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Note that $\alpha + \beta = 1$ so $1 + (-\alpha) = \beta$ and $1 + (-\beta) = \alpha$. Thus, by the Binomial Theorem,

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\alpha^k - \beta^k}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \beta^k\end{aligned}$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-\alpha)^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-\beta)^k$$

$$= \frac{1}{\sqrt{5}}(1+(-\alpha))^n - \frac{1}{\sqrt{5}}(1+(-\beta))^n$$

$$= \frac{1}{\sqrt{5}}\beta^n - \frac{1}{\sqrt{5}}\alpha^n$$

$$= -\frac{\alpha^n - \beta^n}{\sqrt{5}} = -F_n.$$

Solution II by the proposers.

Let f be a function such that $f(n) = F_n$. Also, use

$$F_n = F_{n-1} + F_{n-2}$$

to define F_n for all integers n. Now, each of the following can be proved by induction on n.

Let m and n be integers such that $m, n \geq 0$. Then,

$$\triangle^n f(m) = F_{m-n},$$

where

$$\triangle f(x) = f(x+1) - f(x).$$

Let n be an integer such that $n \geq 0$. Then

$$(-1)^n F_{-n} = -F_n.$$

Let n be an integer such that $n \geq 0$. Then

$$\triangle^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$$

Using the first result with $m = 0$ and the last result with $x = 0$, we have

$$F_{-n} = \Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_k.$$

Multiplying by $(-1)^n$ and using the middle result gives

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = -F_n.$$

Generalized Solution III by Don Redmond, Southern Illinois University, Carbondale, Illinois.

We generalize slightly. Let α and β be roots of the quadratic equation

$$x^2 = ax + b,$$

where a and b are given numbers. Let

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

Note that $\alpha + \beta = a$ and $\alpha\beta = -b$.

If $0 \le r \le n$, then

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-1)^k U_{k+r} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-1)^k \frac{\alpha^{k+r} - \beta^{k+r}}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta} \left\{ \alpha^r \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-1)^k \alpha^k - \beta^r \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-1)^k \beta^k \right\}$$

$$= \frac{1}{\alpha - \beta} \left\{ \alpha^r (a - \alpha)^n - \beta^r (a - \beta)^n \right\} = \frac{1}{\alpha - \beta} \left\{ \alpha^r \beta^n - \alpha^n \beta^r \right\}$$

$$= \frac{1}{\alpha - \beta} (\alpha\beta)^r \left\{ \beta^{n-r} - \alpha^{n-r} \right\} = -(-b)^r U_{n-r}.$$

If $a = b = 1$, then $U_n = F_n$, and so we have, if $r = 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = -F_n.$$

In a similar way, one can show that if $0 \le r \le n$, then

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-1)^k V_{k+r} = (-b)^r V_{n-r}$$

and, in particular,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k = L_n.$$

Comment by Gerald Bergum, South Dakota State University, Brookings, South Dakota.

The problem is a direct result of the identity

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{i+j} = (-1)^{j+1} F_{n-j},$$

which can be found on page 51 of the book *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr.

95. [1996, 90] *Proposed by Stanley Rabinowitz, Westford, Massachusetts and Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

Let n be a positive integer. It is known that

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} 1 = \phi(n),$$

where ϕ is Euler's phi function.

(a) Prove

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k = \frac{1}{2} n \cdot \phi(n).$$

(b) Prove

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k^2 = \frac{1}{6} n\big(2n \cdot \phi(n) + \phi^{-1}(n)\big),$$

where ϕ^{-1} is the Dirichlet inverse of Euler's phi function.

(c)* Let m be a positive integer, $m \geq 3$. Find a formula for

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k^m.$$

Solution by Thomas C. Leong, The City College of City University of New York, New York, New York.

(a) Now $(k,n)=1$ if and only if $(n-k,n)=1$. Thus,

$$2 \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} k = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} k + \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} (n-k) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} n = n \cdot \phi(n)$$

and the result follows.

(b) and (c) We show that

$$\phi_m(n) := \sum_{\substack{1\le k\le n\\(k,n)=1}} k^m = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m-j+1}\prod_{p|n}(1-p^{j-1}), \tag{1}$$

where B_j denote the jth Bernoulli number (see, for example, Kenneth Ireland and Michael Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer-Verlag, New York, 1990.) To prove this, we need several preliminary lemmas.

The first lemma is the well-known formula for summing the mth powers of the first $n-1$ natural numbers. We state it without proof. (For a proof, see Ireland and Rosen.)

Lemma 1. Let $S_m(n) = 1^m + 2^m + \cdots + n^m$. For $n \ge 1$ and $m \ge 1$

$$S_m(n-1) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m-j+1}. \tag{2}$$

The similarity between (1) and (2) is not a coincidence. Their connection is given in the next lemma.

Lemma 2.

$$\sum_{d|n}\phi_m(d)(n/d)^m = S_m(n).$$

Proof. Each fraction in the set $H = \{h/n \mid 1 \le h \le n\}$ can be uniquely expressed in lowest terms: $h/n = a/d$ where $d|n$, $1 \le a \le d$, and $(a,d) = 1$. That is, each h, $1 \le h \le n$, has a unique representation as

$$h = a\frac{n}{d} \quad \text{where } d|n,\ 1 \le a \le d,\ \text{and } (a,d) = 1.$$

Conversely, each fraction a/d where $d|n$, $1 \le a \le d$, and $(a,d) = 1$ is a member of $H:\ a/d = h/n$ where $1 \le h \le n$. That is, if $d|n$, $1 \le a \le d$, and $(a,d) = 1$, then

$$a\frac{n}{d} = h \quad \text{where } 1 \le h \le n.$$

Thus,

$$\sum_{d|n} \phi_m(d)\left(\frac{n}{d}\right)^m = \sum_{d|n} \sum_{\substack{1\le a\le d\\(a,d)=1}} a^m \left(\frac{n}{d}\right)^m = \sum_{d|n} \sum_{\substack{1\le a\le d\\(a,d)=1}} \left(a\frac{n}{d}\right)^m = \sum_{h=1}^{n} h^m = S_m(n).$$

Let $f_m(n) = n^m$. The previous lemma, stated concisely, says $\phi_m * f_m = S_m$, where $*$ denotes Dirichlet convolution. Thus, if we know the Dirichlet inverse f_m^{-1} of f_m, we can find $\phi_m(n)$ by inverting:

$$\phi_m = \phi_m * f_m * f_m^{-1} = S_m * f_m^{-1}.$$

Two more lemmas help us in this computation.

<u>Lemma 3</u>. Let $f_m(n) = n^m$ and $\mu(n)$ denote the Möbius function. Then,

$$f_m^{-1}(n) = \mu(n) f_m(n).$$

<u>Proof</u>. Indeed,

$$(f^{-1} * f)(n) = \sum_{d|n} \mu(d) f_m(d) f_m\left(\frac{n}{d}\right) = n^m \sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1, \end{cases}$$

where we have used the well-known formula

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases} \tag{3}$$

<u>Lemma 4</u>. For any integer m,

$$\sum_{d|n} d^m \mu(d) = \prod_{p|n} (1 - p^m).$$

Proof. Let $g(n) = \sum_{d|n} d^m \mu(d)$. Suppose $(n_1, n_2) = 1$. Then $d|n_1 n_2$ if and only if $d = d_1 d_2$, $d_1|n_1$, and $d_2|n_2$. Thus,

$$\begin{aligned} g(n_1)g(n_2) &= \left(\sum_{d_1|n_1} d_1 \mu(d_1)\right)\left(\sum_{d_2|n_2} d_2 \mu(d_2)\right) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} d_1 d_2 \mu(d_1)\mu(d_2) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} d_1 d_2 \mu(d_1 d_2) \\ &= \sum_{d|n_1 n_2} d\mu(d) \\ &= g(n_1 n_2). \end{aligned}$$

That is, g is multiplicative. Thus, to find $g(n)$ we need only know the values $g(p^a)$ for prime powers p^a. Since $\mu(p^a) = 0$ for $a > 1$,

$$g(p^a) = \sum_{d|p^a} d^m \mu(d) = 1^m \mu(1) + p^m \mu(p) = 1 - p^m$$

and consequently

$$g(n) = \prod_{p|n} g(p^a) = \prod_{p|n} (1 - p^m).$$

Finally, we invert. If $n = 1$, then $\phi_m(n) = 1$. So suppose $n > 1$. Using our previous lemmas,

$$
\begin{aligned}
\phi_m(n) &= (S_m * f_m^{-1})(n) \\
&= \sum_{d|n} S_m(d)\mu\left(\frac{n}{d}\right) f_m\left(\frac{n}{d}\right) \\
&= \sum_{d|n} S_m(d)\frac{n^m}{d^m}\mu\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \left\{ S_m(d-1)\frac{n^m}{d^m}\mu\left(\frac{n}{d}\right) + n^m\mu\left(\frac{n}{d}\right) \right\} \\
&= \sum_{d|n} S_m(d-1)\frac{n^m}{d^m}\mu\left(\frac{n}{d}\right) + n^m \sum_{d|n}\mu\left(\frac{n}{d}\right) \\
&= \sum_{d|n} S_m(d-1)\frac{n^m}{d^m}\mu\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j d^{m-j+1}\frac{n^m}{d^m}\mu\left(\frac{n}{d}\right) \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\sum_{d|n}\binom{m+1}{j}B_j n^m\frac{1}{d^{j-1}}\mu\left(\frac{n}{d}\right) \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m-j+1}\sum_{d|n}\left(\frac{n}{d}\right)^{j-1}\mu\left(\frac{n}{d}\right) \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m-j+1}\prod_{p|n}(1-p^{j-1}).
\end{aligned}
$$

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$. Applying our formula in the case $m = 1, 2, 3, 4$ gives

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} k = \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{1}{2} n \phi(n)$$

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} k^2 = \frac{1}{3} n^3 \prod_{p|n} \left(1 - \frac{1}{p}\right) + \frac{1}{6} n \prod_{p|n} (1-p) = \frac{1}{3} n^2 \phi(n) + \frac{1}{6} n \prod_{p|n} (1-p)$$

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} k^3 = \frac{1}{4} n^4 \prod_{p|n} \left(1 - \frac{1}{p}\right) + n^2 \prod_{p|n} (1-p) = \frac{1}{4} n^3 \phi(n) + n^2 \prod_{p|n} (1-p)$$

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} k^4 = \frac{1}{5} n^5 \prod_{p|n} \left(1 - \frac{1}{p}\right) + \frac{1}{3} n^3 \prod_{p|n} (1-p) - \frac{1}{30} n \prod_{p|n} (1-p^3)$$

$$= \frac{1}{5} n^4 \phi(n) + \frac{1}{3} n^3 \prod_{p|n} (1-p) - \frac{1}{30} n \prod_{p|n} (1-p^3).$$

Note that inverting the well-known formula $\sum_{d|n} \phi(d) = n$, we get $\phi = \mu * f_1$. Thus, $\phi^{-1} = \mu^{-1} * f_1^{-1}$. Now from (3), we see that $\mu^{-1}(n) = 1$ for all n. Applying lemmas 3 and 4, we have

$$\phi^{-1}(n) = \sum_{d|n} f_1^{-1}(d) = \sum_{d|n} d\mu(d) = \prod_{p|n} (1-p).$$

Thus, the above formula for $\phi_2(n)$ and the one in the statement of the problem agree.

Also solved by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri; Don Redmond, Southern Illinois University, Carbondale, Illinois; and James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri. Parts (a) and (b) were also solved by the proposers.

96. [1996, 91] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let an (m by 2) determinant

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m-1} & a_{2,m} \end{pmatrix}, \quad m \geq 3$$

be defined by $S(m-1) + \overline{D}$, where

$$S(m-1) = \sum_{r=1}^{m-1} \det\begin{pmatrix} a_{1,r} & a_{1,r+1} \\ a_{2,r} & a_{2,r+1} \end{pmatrix}$$

and

$$\overline{D} = \det\begin{pmatrix} a_{1,m} & a_{1,1} \\ a_{2,m} & a_{2,1} \end{pmatrix}.$$

Consider an (m by 2) determinant D whose elements are polygonal numbers $P_{n,k}$ ($P_{5,3}$ is the 5th triangular number) and such that

$$a_{i,j} = P_{j,i+2},$$

where $i = 1, 2$ and $j = 1, 2, 3, \ldots, m$. Evaluate D.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.

<u>Solution</u>.

$$D = \frac{m(m-1)(m-2)}{6}.$$

<u>Proof</u>.

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m-1} & a_{2,m} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 3 & 6 & \cdots & \frac{m(m-1)}{2} & \frac{m(m+1)}{2} \\ 1 & 4 & 9 & \cdots & (m-1)^2 & m^2 \end{pmatrix},$$

so for all r, $1 \le r \le m-1$,

$$\det\begin{pmatrix} a_{1,r} & a_{1,r+1} \\ a_{2,r} & a_{2,r+1} \end{pmatrix} = \det\begin{pmatrix} \frac{r(r+1)}{2} & \frac{(r+1)(r+2)}{2} \\ r^2 & (r+1)^2 \end{pmatrix} = \frac{r(r+1)^3}{2} - \frac{r^2(r+1)(r+2)}{2}$$

$$= \frac{r(r+1)}{2}\Big[r^2+2r+1-r^2-2r\Big] = \frac{r(r+1)}{2}.$$

Then,

$$S(m-1) = \sum_{r=1}^{m-1} \det\begin{pmatrix} a_{1,r} & a_{1,r+1} \\ a_{2,r} & a_{2,r+1} \end{pmatrix} = \sum_{r=1}^{m-1} \frac{r(r+1)}{2}$$

$$= \frac{1}{2}\sum_{r=1}^{m-1}(r^2+r) = \frac{1}{2}\left(\frac{m(m-1)(2m-1)}{6} + \frac{m(m-1)}{2}\right)$$

$$= \frac{1}{2}\frac{m(m-1)}{2}\left(\frac{2m-1+3}{3}\right) = \frac{m(m-1)(m+1)}{6}.$$

$$\overline{D} = \det\begin{pmatrix} a_{1,m} & a_{1,1} \\ a_{2,m} & a_{2,1} \end{pmatrix} = \det\begin{pmatrix} \frac{m(m+1)}{2} & 1 \\ m^2 & 1 \end{pmatrix}$$

$$= \frac{m(m+1)}{2} - m^2 = -\frac{m(m-1)}{2}.$$

Thus,

$$S(m-1) + \overline{D} = \frac{m(m-1)(m+1)}{6} - \frac{m(m-1)}{2} = \frac{m(m-1)}{2}\left(\frac{m-2}{3}\right),$$

so that

$$D = \frac{m(m-1)(m-2)}{6}.$$

Also solved by the proposer.
/bye /bye