EVALUATING A FAMILY OF INTEGRALS

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In the study of blackbody radiation, a relationship between the Stefan–Boltzmann constant and Planck's constant can be derived using the fact that

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}. \tag{1}$$

The purpose of this paper is to obtain a generalization of identity (1).

Consider the family of improper integrals defined by

$$I(p) = \int_0^\infty \frac{x^p dx}{e^x - 1}.$$

Conditions will be imposed on p to ensure the convergence of the integral. If $p \geq 1$, then $f(x) = x^p/(e^x - 1)$ has a removable singularity at $x = 0$. Now,

$$\frac{1}{e^x - 1} = \frac{1}{e^x(1 - e^{-x})} = e^{-x} \sum_{n=0}^\infty (e^{-x})^n$$

for $e^{-x} < 1$. So, for $x > 0$,

$$\frac{1}{e^x - 1} = \sum_{n=0}^\infty e^{-(n+1)x}$$

with the convergence being uniform on compact subsets of the interval of convergence. Hence, if $p > 0$, then

$$\frac{x^p}{e^x - 1} = \sum_{n=0}^\infty x^p e^{-(n+1)x}$$

for $x \geq 0$. From the definition of the Gamma function [4], it is known that

$$\int_0^\infty x^p e^{-(n+1)x} dx = \frac{\Gamma(p+1)}{(n+1)^{p+1}}.$$

Therefore, for $p > 0$,

$$I(p) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{1}{(n+1)^{p+1}}$$

$$= \Gamma(p+1) \sum_{n=1}^{\infty} \frac{1}{n^{p+1}}. \tag{2}$$

If p is an odd positive integer, then the series in (2) converges and has been evaluated in [1] using various expansion techniques beginning with the logarithmic derivative of the infinite product expansion of $\sin x$. In this paper, the series in (2), with p an odd positive integer, will be evaluated in closed form using residue theory. To this end, consider

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} = \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' \frac{1}{n^{p+1}}$$

where the prime attached to the summation indicates that the term corresponding to $n = 0$ is to be omitted. It has been shown in [3] that if $f(z)$ satisfies

$$|f(z)| \leq \frac{M}{|z|^k}$$

on C_N for all nonnegative integers N where $k > 1$ and M are constants independent of N, and C_N is the square with vertices at $(N + 1/2)(\pm 1 \ \pm i)$, then

$$\sum_{n=-\infty}^{\infty} f(n)$$

is the negative of the sum of the residues of $\pi f(z)\cot\pi z$ at the poles of $f(z)$. So, to evaluate the summation in (2), take

$$f(z) = \frac{1}{z^{p+1}}$$

and use $\sum'$.

It is known from [2] that

$$z\cot z = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} z^{2n}}{(2n)!}$$

for $|z| < \pi$, where the B_{2n}'s are the Bernoulli numbers of even index. Hence,

$$\frac{\pi\cot\pi z}{z^{p+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n} B_{2n} z^{2n-p-2}}{(2n)!}$$

for $0 < |z| < 1$. So,

$$\mathrm{Res}_{z=0}\frac{\pi\cot\pi z}{z^{p+1}} = \frac{(-1)^{(p+1)/2}(2\pi)^{p+1}B_{p+1}}{(p+1)!}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^{p+1}} = -\frac{(-1)^{(p+1)/2}(2\pi)^{p+1}B_{p+1}}{2(p+1)!}.$$

Substituting the latter identity into (2) and simplifying yields

$$\begin{aligned} I(p) &= \frac{(-1)^{(p+3)/2}(2\pi)^{p+1}B_{p+1}}{2(p+1)} \\ &= \frac{(-1)^{(p+3)/2}2^p\pi^{p+1}B_{p+1}}{p+1}. \end{aligned} \tag{3}$$

In conclusion, identity (3) is valid for all odd positive integers p and provides a generalization of (1). In particular, for $p = 3$, equation (3) becomes

$$I(3) = -2\pi^4 B_4 = -2\pi^4(-1/30) = \pi^4/15,$$

which agrees with (1).

References

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