TWO SOLUTIONS OF ONE PROBLEM OF B. S. POPOV

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In this paper are given two solutions of the problem of Prof. B. S. Popov [1].

1. The Problem of Prof. B. S. Popov. In [1], Prof. Popov formulates the following problem.

Prove that the determinant $D_n = |a_{ij}|$, where

$$\begin{aligned} &a_{ij} = 0, \quad (j = i + s, s > 1); \quad a_{ij} = 1, \quad (j = i + 1); \\ &a_{ij} = a, \quad (i = j) \ \text{ and } \ a_{ij} = 2i - 2, \quad (j = i - 1), \end{aligned}$$

satisfies the following identities:

$$(i) \quad D_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \cdot \frac{n!}{k!} \cdot \frac{a^{n-2k}}{(n-2k)!},$$

$$(ii) \quad D_n^2 = 2^n \cdot n! \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{D_{2k}}{2^k \cdot k!}.$$

2. First Solution of the Problem. The given determinant is

$$D_n = \begin{vmatrix} a & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & a & 1 & \cdots & 0 & 0 & 0 \\ 0 & 4 & a & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a & 1 & 0 \\ 0 & 0 & 0 & \cdots & 2n-4 & a & 1 \\ 0 & 0 & 0 & \cdots & 0 & 2n-2 & a \end{vmatrix}.$$

Developing it by the last column, we obtain the recurrent formula

$$D_n = aD_{n-1} - (2n-2)D_{n-2}. \tag{1}$$

Note that $D_1 = a$ and $D_2 = a^2 - 2$, and putting $D_0 = 1$ and $D_{-1} = 0$, we obtain that (1) holds for $n \geq 1$. The proof of (i) and (ii) is by induction of n, and (i) will not be used in the proof of (ii).

Proof of (i). It is easy to verify that (i) holds for $n = 1$ and $n = 2$. Assume that (i) holds for the numbers $n-1$ and $n-2$, $(n > 2)$. Then

$$D_n = aD_{n-1} - (2n-2)D_{n-2} = a \cdot \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot \frac{a^{n-2k-1}}{(n-2k-1)!}$$

$$- (2n-2) \cdot \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \cdot \frac{(n-2)!}{k!} \cdot \frac{a^{n-2k-2}}{(n-2k-2)!}$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot \frac{a^{n-2k}}{(n-2k-1)!}$$

$$- 2 \cdot \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot \frac{a^{n-2k-2}}{(n-2k-2)!}$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot \frac{a^{n-2k}}{(n-2k-1)!}$$

$$+ 2 \cdot \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \cdot \frac{(n-1)!}{(k-1)!} \cdot \frac{a^{n-2k}}{(n-2k)!}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot \frac{a^{n-2k}}{(n-2k)!}[(n-2k) + 2k]$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \cdot \frac{n!}{k!} \cdot \frac{a^{n-2k}}{(n-2k)!},$$

and (i) is proved.

Proof of (ii). In order to prove (ii) by induction of n, D_n^2 should be expressed by D_{n-1}^2, D_{n-2}^2, ... analogously as in (1). Indeed, we will prove the following identity.

(2) $D_n^2 = (a^2 - 2n + 2)D_{n-1}^2 - (2n-2)(a^2 - 2n + 2)D_{n-2}^2 + (2n-2)(2n-4)^2 D_{n-3}^2$,

for $n \geq 4$.

The identity (1) implies

$$D_n^2 = a^2 D_{n-1}^2 + 4(n-1)^2 D_{n-2}^2 - 2(2n-2)aD_{n-1}D_{n-2}.$$

On the other side,

$$aD_{n-1}D_{n-2} = D_{n-1}[D_{n-1} + (2n-4)D_{n-3}] = D_{n-1}^2 + (2n-4)D_{n-1}D_{n-3}.$$

The identity

$$D_{n-1} + (2n-4)D_{n-3} = aD_{n-2}$$

implies

$$D_{n-1}^2 + (2n-4)^2 D_{n-3}^2 + 2(2n-4)D_{n-1}D_{n-3} = a^2 D_{n-2}^2,$$

$$(2n-4)D_{n-1}D_{n-3} = \frac{1}{2}[a^2 D_{n-2}^2 - D_{n-1}^2 - (2n-4)^2 D_{n-3}^2].$$

Hence,

$$aD_{n-1}D_{n-2} = D_{n-1}^2 + \frac{1}{2}[a^2 D_{n-2}^2 - D_{n-1}^2 - (2n-4)^2 D_{n-3}^2]$$

$$= \frac{1}{2}[a^2 D_{n-2}^2 + D_{n-1}^2 - (2n-4)^2 D_{n-3}^2],$$

$$D_n^2 = a^2 D_{n-1}^2 + 4(n-1)^2 D_{n-2}^2 - (2n-2)[a^2 D_{n-2}^2 + D_{n-1}^2 - (2n-4)^2 D_{n-3}^2]$$
$$= (a^2 - 2n + 2)D_{n-1}^2 - (2n-2)(a^2 - 2n + 2)D_{n-2}^2 + (2n-2)(2n-4)^2 D_{n-3}^2$$

and (2) is proved. We should also prove the following identity.

$$a^2 D_{n-2} = D_n + (4n-6)D_{n-2} + (2n-4)(2n-6)D_{n-4}, \tag{3}$$

for $n \geq 5$. Indeed,

$$\begin{aligned}
D_n &= aD_{n-1} - (2n-2)D_{n-2} = a[aD_{n-2} - (2n-4)D_{n-3}] - (2n-2)D_{n-2} \\
&= a^2 D_{n-2} - (2n-2)D_{n-2} - (2n-4)aD_{n-3} \\
&= a^2 D_{n-2} - (2n-2)D_{n-2} - (2n-4)[D_{n-2} + (2n-6)D_{n-4}] \\
&= a^2 D_{n-2} - (4n-6)D_{n-2} - (2n-4)(2n-6)D_{n-4}
\end{aligned}$$

and (3) is proved.

The identity (3) implies the following identity.

$$\begin{aligned}
D_{2k}(a^2 - 2n + 2) &= D_{2(k+1)} + (8k+2)D_{2k} + 4k(4k-2)D_{2(k-1)} - (2n-2)D_{2k} \\
&= D_{2(k+1)} + (8k+4-2n)D_{2k} + 4k(4k-2)D_{2(k-1)}.
\end{aligned}$$

Using this identity and (3), we are able to prove (ii).

It is easy to verify (ii) for $n = 1, 2, 3, 4$. Assume that (ii) holds for the numbers $n-1$, $n-2$ and $n-3$, $(n \geq 5)$. Let us denote

$$A_j^i = \binom{i}{j} \cdot 2^{i-j} \cdot \frac{i!}{j!}$$

for $i \in \mathbb{N}$ and $0 \leq j \leq i$. By algebraic transformations, the following identity

$$\begin{aligned}
&A_{k-1}^{n-1} + (8k+4-2n)A_k^{n-1} + 4(k+1)(4k+2)A_{k+1}^{n-1} \\
&-(2n-2)A_{k-1}^{n-2} - (2n-2)(8k+4-2n)A_k^{n-2} - (2n-2)4(k+1)(4k+2)A_{k+1}^{n-2} \\
&+ (2n-2)(2n-4)^2 A_k^{n-3} = A_k^n
\end{aligned}$$

can be proved. Indeed, multiplying it by

$$\frac{[(k+1)!]^2 \cdot (n-k)!}{[(n-2)!]^2} \cdot 2^{k-n}$$

one obtains polynomial equality of n and k, and by direct calculation it can be verified. Using this identity, the inductive assumptions and the identity (2), we obtain

$$D_n^2 = (a^2 - 2n + 2)D_{n-1}^2 - (2n-2)(a^2 - 2n + 2)D_{n-2}^2 + (2n-2)(2n-4)^2 D_{n-3}^2$$

$$= \sum_{k=0}^{n-1} A_k^{n-1} D_{2k}(a^2 - 2n + 2) - \sum_{k=0}^{n-2} A_k^{n-2} D_{2k}(a^2 - 2n + 2)(2n-2)$$

$$+ \sum_{k=0}^{n-3} A_k^{n-3} D_{2k}(2n-2)(2n-4)^2$$

$$= \sum_{k=0}^{n-1} A_k^{n-1}[D_{2(k+1)} + (8k + 4 - 2n)D_{2k} + 4k(4k-2)D_{2(k-1)}]$$

$$- \sum_{k=0}^{n-2} A_k^{n-2}(2n-2)[D_{2(k+1)} + (8k + 4 - 2n)D_{2k} + 4k(4k-2)D_{2(k-1)}]$$

$$+ \sum_{k=0}^{n-3} A_k^{n-3}(2n-2)(2n-4)^2 D_{2k}$$

$$= \sum_{k=0}^{n} D_{2k}[A_{k-1}^{n-1} + (8k+4-2n)A_k^{n-1} + 4(k-1)(4k+2)A_{k+1}^{n-1}$$

$$- (2n-2)A_{k-1}^{n-2} - (2n-2)(8k+4-2n)A_k^{n-2} - (2n-2)4(k+1)(4k+2)A_{k+1}^{n-2}$$

$$+ A_k^{n-3}(2n-2)(2n-4)^2] = \sum_{k=0}^{n} A_k^n \cdot D_{2k}.$$

Hence (ii) is proved.

3. Second Solution of the Problem. Now we will prove the identities (i) and (ii) using the properties of the Hermite polynomials.

(i) In the theory of the special functions [2], the Hermite polynomial

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!(2x)^{n-2k}}{k!(n-2k)!}, \tag{4}$$

of nth degree satisfies the following equality

$$H_n(x) = 2xH_{n-1}(x) - (2n-2)H_{n-2}(x). \tag{5}$$

Since (4) is uniquely determined by (5) with the initial conditions $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$, by putting $x = a/2$ from (1), (4) and (5) we obtain

$$D_n = H_n(a/2) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!a^{n-2k}}{k!(n-2k)!},$$

and (i) is proved.

(ii) Starting from the known identity of E. Feldheim [3],

$$H_m(x)H_n(x) = 2^n n! \sum_{k=0}^{n} \binom{m}{n-k} \frac{1}{2^k k!} H_{m-n+2k}(x), \quad (m \geq n) \tag{6}$$

for product of Hermite polynomials, for $m = n$ we obtain

(7) $$H_n^2(x) = 2^n n! \sum_{k=0}^{n} \binom{n}{n-k} \frac{1}{2^k k!} H_{2k}(x).$$

Since D_n satisifies the recurrent formula (1), and also (5) for $x = a/2$, then from (7) it follows

$$D_n^2 = H_n^2(a/2) = 2^n n! \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k k!} D_{2k},$$

and (ii) is proved.

Remark. There are more forms for the product of the Hermite polynomials [4,5,6,7,8,9], and hence, different proofs for (ii). For example using the equation

$$H_m(x)H_n(x) = \sum_{r=0}^{n} 2^r r! \binom{m}{r} \binom{n}{r} H_{m+n-2r}(x), \quad (m \geq n)$$

of G. N. Watson [4], for $m = n$ we obtain

(8) $$D_n^2 = H_n^2(a/2) = \sum_{r=0}^{n} 2^r r! \binom{n}{r}^2 D_{2n-2r}.$$

Exchanging $n - r$ by k in the left side of (8), we obtain (ii).

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