PRESENTATIONS OF SUBGROUPS OF ARTIN GROUPS

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Abstract. Let $A\Gamma$ be the Artin group based on the graph Γ, and let $\phi: A\Gamma \to \mathbb{Z}$ be a homomorphism which maps each of the standard generators of $A\Gamma$ to 0 or 1. We compute an explicit presentation for $\ker \phi$ in the general case. In the case where Γ is a tree with a connected and dominating live subgraph, we prove $\ker \phi$ is a free group and we calculate its rank. In addition, if $A\Gamma$ is a 2–cone with live apex, we prove $\ker \phi$ is isomorphic to the Artin group on the base of the cone, and if Γ is a special tree–triangle combination, we determine conditions on Γ which ensure the finite presentation of $\ker \phi$.

1. Introduction. To each finite simple graph Γ whose edges are weighted with integers greater than 1, we can associate a group $A\Gamma$ with generators in one–to–one correspondence with the vertices of Γ and relations $[x,y]_k = [y,x]_k$ for each edge $\{x,y\}$ of weight k, where $[x,y]_k = \underbrace{xyx\ldots}_{k \text{ letters}}$. Such groups are known as Artin groups and Γ is the defining graph for $A\Gamma$. The class of Artin groups includes the braid groups, the fundamental groups of $(2,n)$–torus link complements, and graph groups (all edges labelled 2), which include all finitely generated free and free abelian groups. Kernels of homomorphisms from graph groups to $\mathbb{Z}$ provided the first examples of groups of type FP_n but not type FP_{n+1}, as well as groups of type FP_2 which are not finitely presented [1].

In this paper, we calculate presentations of kernels of homomorphisms from Artin groups onto $\mathbb{Z}$, where each generator is mapped to 0 or 1. We apply the methods similar to those employed in [3] to this situation. The cases we consider are Artin groups based on cones, trees, and combinations thereof.

The primary results of this paper consist of three theorems. The first of these allows us to find an explicit, finite presentation for the kernels of the above types of epimorphisms from Artin groups based on trees onto $\mathbb{Z}$.

<u>Theorem 1</u>. Let T be a finite, weighted tree and let $\phi: AT \to \langle t \rangle$ be the epimorphism which sends each generator to t or 1. If $\mathcal{L}(\phi)$ is connected and dominating,

then $\ker\phi$ is the free group on $N = \sum_{e_i \in \mathcal{L}(\phi)} (W(e_i) - 1) + \sum_{e_j \notin \mathcal{L}(\phi)} \frac{W(e_j)}{2}$ generators, where $W(e)$ represents the weight of edge e.

Our next result provides sufficient conditions for the kernels of certain epimorphisms from Artin groups onto $\mathbb{Z}$ to be finitely presented.

Theorem 2. Let Γ be a finite, simple, weighted graph whose vertices are partitioned into sets $L = \{l_1, l_2, \ldots, l_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$. Let $\phi: A\Gamma \to \langle t \rangle$ be an epimorphism defined by $\phi(l_i) = t$ and $\phi(d_i) = 1$ for all i. Then $\ker\phi$ is finitely presented if there exists a sequence of subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of Γ where Γ_1 consists of a single vertex of L, $\Gamma_n = \Gamma$, and Γ_{j+1} can be obtained from Γ_j by either

1. adding a vertex v and an edge $\{v, l_i\}$, where $l_i \in V(\Gamma_j)$, or
2. adding an edge $\{a, b\}$, where $a, b, l_i \in V(\Gamma_j)$; $\{a, l_i\}, \{b, l_i\} \in E(\Gamma_j)$; and $k_{\{a,l_i\}} = k_{\{b,l_i\}} = 2$.

Our third theorem concerns Artin groups based on cones and gives an explicit, finite presentation for the kernels of certain epimorphisms from such Artin groups onto $\mathbb{Z}$. In this case, the kernel is itself an Artin group.

Theorem 3. Let $A\Gamma$ be an Artin group where Γ is a 2–cone with apex a and base Γ_1. If $\phi: A\Gamma \to \langle t \rangle$ is an epimorphism mapping all generators to either t or 1 with $\phi(a) = t$, then $\ker\phi \simeq A\Gamma_1$. In particular, $\ker\phi$ is finitely presented.

2. Preliminaries. We assume the reader is familiar with the concept of a graph. A simple graph is a graph with no loops or multiple edges. Given a simple graph, Γ, we will denote the set of vertices of Γ by $V(\Gamma)$ and the set of edges, consisting of unordered pairs of elements of $V(\Gamma)$, by $E(\Gamma)$. Γ is finite if both $V(\Gamma)$ and $E(\Gamma)$ are finite. A weighted simple graph is a simple graph Γ for which each edge $\{v, w\} \in E(\Gamma)$ is labelled by an integer $k = k_{\{v,w\}} > 1$.

A finite, simple, weighted graph Γ induces a presentation of a group

$$A\Gamma = \langle V(\Gamma) | [x, y]_k = [y, x]_k \text{ for all } \{x, y\} \in E(\Gamma) \rangle,$$

where $[x, y]_k = \underbrace{xyx\ldots}_{k \text{ letters}}$. A group G is called an Artin group provided there exists some finite, simple, weighted graph Γ such that $G \simeq A\Gamma$. Given any Artin group, we will assume the above presentation and identify the set of generators of $A\Gamma$ with the vertex set $V(\Gamma)$.

Following [5], we will denote the Tietze transformations adjunction of relators, deletion of relators, adjunction of generators, and deletion of generators by (T1), (T2), (T3), and (T4), respectively. It is shown in [5] that given any two presentations of a group G, one can be obtained from the other by repeated applications of these four transformations.

Given a presentation of a group G and suitable information about a subgroup $H \leq G$, the Reidemeister–Schreier method enables one to obtain a presentation for H. In [4], it is shown that if $\langle X|R\rangle$ is a presentation for G, $\pi\colon F(X) \to G$ is the canonical homomorphism, T is a Schreier transversal for $\pi^{-1}(H)$ in $F(X)$, and $\Phi\colon F(X) \to T$ is the function which maps each element to its coset representative, then $Y = \{tx\Phi(tx)^{-1} : t \in T, x \in X, tx \notin T\}$ is a set of generators for H. Furthermore, if $\tau\colon \pi^{-1}(H) \to F(Y)$ is the function in [4] which rewrites each $w \in \pi^{-1}(H)$ in terms of the generators Y and $S = \{\tau(trt^{-1}) : t \in T, r \in R\}$, then $\langle Y|S\rangle$ is a presentation for H.

The reader is directed to [4] or [5] for details of Tietze transformations and the Reidemeister–Schreier rewriting procedure.

3. The Subgroups. Let Γ be a finite, simple, weighted graph and $A\Gamma$ its corresponding Artin group. Partition $V(\Gamma)$ into two sets, $L = \{l_0, l_1, \dots, l_r\}$ and $D = \{d_1, d_2, \dots, d_s\}$, and define the homomorphism $\phi\colon A\Gamma \to \langle t\rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ for all i. Following [6], we will refer to the vertices in L as live, and denote the full subgraph spanned by L as $\mathcal{L}(\phi)$, the living subgraph of Γ. We will refer to the vertices in D as dead.

For a Schreier transversal for $\pi^{-1}(\ker\phi)$ in $F(V(\Gamma))$, take $T = \{l_0^n : n \in \mathbb{Z}\}$ and recall that $\Phi\colon F(V(\Gamma)) \to T$ sends each element to its coset representative. A set of generators for $\ker\phi$ is then $\{tx\Phi(tx)^{-1} : t \in T, x \in V(\Gamma), tx \notin T\}$, which gives the families of generators

$$\begin{aligned} \lambda(i,n) &= l_0^n l_i \Phi(l_0^n l_i)^{-1} = l_0^n l_i l_0^{-(n+1)} \\ \text{and } \delta(j,n) &= l_0^n d_j \Phi(l_0^n d_j)^{-1} = l_0^n d_j l_0^{-n}, \end{aligned}$$

where $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$, and $n \in \mathbb{Z}$. We obtain the relations by rewriting the set $\{trt^{-1} : t \in T, r \in R\}$ in terms of these generators, where

$$R = \{[x,y]_k[y,x]_k^{-1} : \{x,y\} \in E(\Gamma) \text{ and } k = k_{\{x,y\}}\}.$$

For convenience, we define $\lambda(0,n) = 1$ for all $n \in \mathbb{Z}$.

The five cases for the relations $r \in R$ yield the following presentation for $\ker\phi$:

- **Generators**:

$$\begin{cases} \lambda(i,n), & 1 \leq i \leq r,\ n \in \mathbb{Z} \\ \delta(j,n), & 1 \leq j \leq s,\ n \in \mathbb{Z}. \end{cases}$$

- **Relations**:

– For each $\{l_i, l_j\} \in E(\Gamma)$ with k even:

(1) $$\lambda(i,n)\lambda(j,n+1)\lambda(i,n+2)\cdots\lambda(i,n+k-2)\lambda(j,n+k-1) = \lambda(j,n)\lambda(i,n+1)\lambda(j,n+2)\cdots\lambda(j,n+k-2)\lambda(i,n+k-1).$$

– For each $\{l_i, l_j\} \in E(\Gamma)$ with k odd:

(2) $$\lambda(i,n)\lambda(j,n+1)\lambda(i,n+2)\cdots\lambda(j,n+k-2)\lambda(i,n+k-1) = \lambda(j,n)\lambda(i,n+1)\lambda(j,n+2)\cdots\lambda(i,n+k-2)\lambda(j,n+k-1).$$

– For each $\{l_i, d_j\} \in E(\Gamma)$ (and k necessarily even):

(3) $$\lambda(i,n)\delta(j,n+1)\lambda(i,n+1)\cdots\lambda\left(i,n+\frac{k}{2}-1\right)\delta\left(j,n+\frac{k}{2}\right) = \delta(j,n)\lambda(i,n)\delta(j,n+1)\cdots\delta\left(j,n+\frac{k}{2}-1\right)\lambda\left(i,n+\frac{k}{2}-1\right).$$

– For each $\{d_i, d_j\} \in E(\Gamma)$ with k even:

(4) $$\underbrace{\delta(i,n)\delta(j,n)\delta(i,n)\cdots\delta(i,n)\delta(j,n)}_{k \text{ terms}} = \underbrace{\delta(j,n)\delta(i,n)\delta(j,n)\cdots\delta(j,n)\delta(i,n)}_{k \text{ terms}}.$$

– For each $\{d_i, d_j\} \in E(\Gamma)$ with k odd:

$$(5) \qquad \underbrace{\delta(i,n)\delta(j,n)\delta(i,n)\cdots\delta(j,n)\delta(i,n)}_{k \text{ terms}} = \underbrace{\delta(j,n)\delta(i,n)\delta(j,n)\cdots\delta(i,n)\delta(j,n)}_{k \text{ terms}}.$$

Note that every element of $V(\Gamma) - \{l_0\}$ corresponds to a countable family of generators and every element of $E(\Gamma)$ corresponds to a countable family of relations.

4. Artin Groups Based on Trees. If T is a tree, it is shown in [2] that the kernel of a map $\phi: AT \to \langle t \rangle$ in which each generator is mapped to t is a free group on $\sum_{e_i \in E(T)} (W(e_i) - 1)$ generators, where $W(e_i)$ is the weight of edge e_i. In this section, we use the presentations computed in the previous section to extend this result to kernels of maps $\phi: AT \to \langle t \rangle$ where each generator is sent to t or 1 in the case that the living subtree is connected and dominating. Recall that a subgraph Γ' of a graph Γ is dominating if each vertex in $\Gamma - \Gamma'$ is adjacent to a vertex in Γ'.

<u>Lemma 1</u>. Let Γ be a finite, simple, weighted graph and $\phi: A\Gamma \to \langle t \rangle$ be an epimorphism sending each generator to t or 1. Let Γ_1 be the graph constructed from Γ by adding vertex v and edge $\{v, l\}$ of weight k with $l \in \mathcal{L}(\phi)$. If $\phi_1: A\Gamma_1 \to \langle t \rangle$ is an epimorphism that agrees with ϕ on $V(\Gamma)$ and maps v to either t or 1, then

$$\ker \phi_1 \simeq \ker \phi * F_a \text{ where } a = \begin{cases} k-1, & \text{if } \phi_1(v) = t \\ \frac{k}{2}, & \text{if } \phi_1(v) = 1. \end{cases}$$

<u>Proof</u>. We will use the presentation for $\ker \phi$ calculated in Section 3 and denote it by $\langle X | R \rangle$. Since exactly one vertex and one edge have been added to Γ, $\ker \phi_1$ may be presented using the generators X and relators R together with one additional family of generators and relators. We denote the family of generators corresponding to the vertex l by λ_n and the family of generators corresponding to v by σ_n for $n \in \mathbb{Z}$.

We first consider the case $\phi_1(v) = t$. With the above notation and a straightforward calculation, family (1) becomes

$$(6) \qquad \sigma_n = \lambda_n \sigma_{n+1} \lambda_{n+2} \sigma_{n+3} \cdots \lambda_{n+k-2} \sigma_{n+k-1} \lambda_{n+k-1}^{-1} \sigma_{n+k-2}^{-1} \cdots \lambda_{n+1}^{-1}.$$

Similarly, family (2) becomes

$$(7) \qquad \sigma_n = \lambda_n \sigma_{n+1} \lambda_{n+2} \sigma_{n+3} \cdots \lambda_{n+k-1} \sigma_{n+k-1}^{-1} \cdots \lambda_{n+1}^{-1}.$$

Straightforward induction shows that for $n < 0$ or $n > k-2$, σ_n can be written in terms of σ_j for $j = 0, 1, \ldots, k-2$ together with elements of X. By (T4), we can eliminate every relation in family (6) or (7) and every generator σ_n for $n < 0$ or $n > k-2$. The generators σ_i do not appear in the presentation for $\ker\phi$, so $\langle X \cup \{\sigma_m\} | R\rangle$ for $m = 0, \ldots, k-2$, is a presentation for $\ker\phi_1$. Thus, $\ker\phi_1 \simeq \ker\phi * F_a$.

We now consider the case where $\phi(v) = 1$ (so k is necessarily even). With the above notation, a straightforward calculation shows family (3) becomes

$$\sigma_n = \lambda_n \sigma_{n+1} \lambda_{n+1} \sigma_{n+2} \cdots \lambda_{n+\frac{k-2}{2}} \sigma_{n+\frac{k}{2}} \lambda^{-1}_{n+\frac{k-2}{2}} \cdots \lambda^{-1}_{n+1}. \tag{8}$$

Straightforward induction shows that for $n < 0$ or $n > \frac{k}{2} - 1$, σ_n can be written in terms of σ_j for $j = 0, 1, \ldots, \frac{k}{2} - 1$ together with elements of X. By (T4), we can eliminate every relation in family (8) and every generator σ_n for $n < 0$ or $n > \frac{k}{2} - 1$. The generators σ_i do not appear in the presentation for $\ker\phi$, so $\langle X \cup \{\sigma_m\} | R\rangle$ for $m = 0, 1, \ldots, \frac{k}{2} - 1$ is a presentation for $\ker\phi_1$. Thus, $\ker\phi_1 \simeq \ker\phi * F_a$.

Theorem 1. Let T be a finite, weighted tree and let $\phi\colon AT \to \langle t\rangle$ be the epimorphism which sends each generator to t or 1. If $\mathcal{L}(\phi)$ is connected and dominating, then $\ker\phi$ is the free group on $N = \sum_{e_i \in \mathcal{L}(\phi)} (W(e_i) - 1) + \sum_{e_j \notin \mathcal{L}(\phi)} \frac{W(e_j)}{2}$ generators, where $W(e)$ represents the weight of edge e.

Proof. The proof will proceed by induction on the number of vertices, n, in T. For $n = 1$, $N = 0$, and $\ker\phi$ is trivial. Suppose $\ker\phi$ is free on N generators for $n \leq m$. Let T have $m+1$ vertices, and let $\hat{T}$ be the tree formed by removing a vertex, v, of valence 1 and the corresponding edge. Since $\mathcal{L}(\phi)$ is connected and dominating, the vertex adjacent to v must be live. (If $n = 2$ and only one vertex is live, take v to be the dead vertex.) So if $\hat{\phi}$ is the epimorphism which agrees with ϕ on $V(\hat{T})$, then by Lemma 1, $\ker\phi \simeq \ker\hat{\phi} * F_{\hat{a}}$ with $\hat{a} = \begin{cases} k-1, & \text{for } v \text{ living} \\ \frac{k}{2}, & \text{for } v \text{ dead.} \end{cases}$

But by hypothesis, $\ker\hat{\phi}$ is free on $\hat{N} = \sum_{e_i \in \mathcal{L}(\hat{\phi})} (W(e_i) - 1) + \sum_{e_j \notin \mathcal{L}(\hat{\phi})} \frac{W(e_j)}{2}$ generators. So $\ker\phi \simeq \ker\hat{\phi} * F_{\hat{a}} \simeq F_N$.

Example 1. Define $\phi: A\Gamma \to \langle t\rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ where Γ is as follows. Then by Theorem 1, $\ker\phi \simeq F_9$.

5. Artin Groups Based on Certain Tree–Triangle Combinations. In this section, we use the tools developed in the previous sections to study kernels of Artin groups which can be constructed by attaching triangles to the living subgraph, provided the previously existing legs both have weight 2.

Lemma 2. Let Γ be a finite, simple, weighted graph and let $\phi: A\Gamma \to \langle t\rangle$ be an epimorphism sending all generators to either t or 1 whose kernel has presentation $\langle X|R\rangle$. Let a, b be vertices of Γ such that there exists a $c \in \mathcal{L}(\phi)$ with $\{a,c\},\{b,c\} \in E(\Gamma)$ and $k_{\{a,c\}} = k_{\{b,c\}} = 2$. Let Γ_1 be the graph constructed from Γ by adding edge $\{a,b\}$, and ϕ_1 be the epimorphism agreeing with ϕ on all the generators of $A\Gamma$. Then $\ker\phi_1 = \langle X|R\cup\{r\}\rangle$ for some relation r.

Proof. We see that vertices a, b, and c are associated with three families of generators for $\ker\phi$ which we will call α_n, β_n, and γ_n, respectively, where $n \in \mathbb{Z}$.

We will first consider the case where $a, b \in \mathcal{L}(\phi)$. Below are the details for the case where k is even; the case for k odd is essentially identical. We see that vertices a, b, and c are associated with the two families of relations

$$\alpha_n = \gamma_n\alpha_{n+1}\gamma_{n+1}^{-1}, \tag{9}$$

$$\beta_n = \gamma_n\beta_{n+1}\gamma_{n+1}^{-1}. \tag{10}$$

The edge $\{a,b\}$ introduces the new relations.

$$\alpha_n\beta_{n+1}\alpha_{n+2}\cdots\alpha_{n+k-2}\beta_{n+k-1} = \beta_n\alpha_{n+1}\beta_{n+2}\cdots\beta_{n+k-2}\alpha_{n+k-1}. \tag{11}$$

However, to derive the entire families, we need only the families (9) and (10) together with the following relation from (11) for $n = 0$:

$$\alpha_0\beta_1\alpha_2\cdots\alpha_{k-2}\beta_{k-1} = \beta_0\alpha_1\beta_2\cdots\beta_{k-2}\alpha_{k-1}. \tag{12}$$

We start with (12) and induct on n in both the positive and negative directions. Suppose that for some $m \in \mathbb{Z}$

$$\alpha_m\beta_{m+1}\alpha_{m+2}\cdots\alpha_{m+k-2}\beta_{m+k-1} = \beta_m\alpha_{m+1}\beta_{m+2}\cdots\beta_{m+k-2}\alpha_{m+k-1} \tag{13}$$

is derivable from (9), (10), and (12). Substituting (9) and (10) into (13), we obtain

$$\alpha_{m+1}\beta_{m+2}\alpha_{m+2}\cdots\alpha_{m+k-1}\beta_{m+k} = \beta_{m+1}\alpha_{m+2}\beta_{m+3}\cdots\beta_{m+k-1}\alpha_{m+k},$$

completing the induction in the positive direction. For the negative direction we again use (9) and (10) to obtain

$$\alpha_{m-1}\beta_m\alpha_{m+1}\cdots\alpha_{m+k-3}\beta_{m+k-2} = \beta_{m-1}\alpha_m\beta_{m+1}\cdots\beta_{m+k-3}\alpha_{m+k-2},$$

completing the induction in the negative direction. We have now shown that the family (11) can be derived from the single relation for $n = 0$. So by (T2), we may delete all relations where $n \neq 0$ from either family. For the case where $a \in \mathcal{L}(\phi)$, $b \notin \mathcal{L}(\phi)$ and the case where $a, b \notin \mathcal{L}(\phi)$, it can be shown by a similar induction that the entire family of relations introduced by edge $\{a, b\}$ can be derived from the single member of the family where $n = 0$. Thus, $\ker\phi_1$ has the presentation $\langle X|R \cup \{r\}\rangle$ for a single relation r.

Theorem 2. Let Γ be a finite, simple, weighted graph whose vertices are partitioned into sets $L = \{l_1, l_2, \ldots, l_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$. Let $\phi\colon A\Gamma \to \langle t\rangle$ be an epimorphism defined by $\phi(l_i) = t$ and $\phi(d_i) = 1$ for all i. Then $\ker\phi$ is finitely presented if there exists a sequence of subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of Γ where Γ_1 consists of a single vertex of L, $\Gamma_n = \Gamma$, and Γ_{j+1} can be obtained from Γ_j by either

1. adding a vertex v and an edge $\{v, l_i\}$, where $l_i \in V(\Gamma_j)$, or
2. adding an edge $\{a, b\}$, where $a, b, l_i \in V(\Gamma_j)$; $\{a, l_i\}, \{b, l_i\} \in E(\Gamma_j)$; and $k_{\{a,l_i\}} = k_{\{b,l_i\}} = 2$.

Proof. For each subgraph Γ_j, define $\phi_j\colon A\Gamma_j \to \langle t\rangle$ to be the restriction of ϕ, and assume $\ker\phi_j$ has presentation $\langle X|R\rangle$. If Γ_{j+1} is constructed from Γ_j by operation 1, then by Lemma 1, $\ker\phi_{j+1}$ has presentation $\langle X \cup \{g_1, g_2, \ldots, g_m\}|R\rangle$ for some m, while if Γ_{j+1} is constructed from Γ_j by operation 2, then by Lemma 2, $\ker\phi_{j+1}$ has the presentation $\langle X|R \cup \{r\}\rangle$. So $\ker\phi_{j+1}$ has only finitely many more generators and relations in its presentation than does $\ker\phi_j$. Thus, $\ker\phi = \ker\phi_n$ has only finitely many more generators and relations than does the trivial group, $\ker\phi_1$.

Example 2. Define $\phi\colon A\Gamma \to \langle t\rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ where Γ is as follows. Note: unlabelled edges have arbitrary weight.

Define Γ_i for $i = 1, 2, \ldots, 9$ as shown in the table below. Then $\Gamma_1, \Gamma_2, \ldots, \Gamma_9$ is a sequence of subgraphs which satisfies the conditions of Theorem 2. Therefore, $\ker \phi$ is finitely presented.

Subgraph	Vertices	Edges
Γ_1	$\{l_1\}$	$\emptyset$
Γ_2	$V(\Gamma_1) \cup \{l_2\}$	$\{\{l_1, l_2\}\}$
Γ_3	$V(\Gamma_2) \cup \{d_1\}$	$E(\Gamma_2) \cup \{\{l_1, d_1\}\}$
Γ_4	$V(\Gamma_3)$	$E(\Gamma_3) \cup \{\{l_2, d_1\}\}$
Γ_5	$V(\Gamma_4) \cup \{l_3\}$	$E(\Gamma_4) \cup \{\{l_1, l_3\}\}$
Γ_6	$V(\Gamma_5)$	$E(\Gamma_5) \cup \{\{l_2, l_3\}\}$
Γ_7	$V(\Gamma_6) \cup \{d_3\}$	$E(\Gamma_6) \cup \{\{d_3, l_3\}\}$
Γ_8	$V(\Gamma_7) \cup \{d_2\}$	$E(\Gamma_7) \cup \{\{l_3, d_2\}\}$
$\Gamma_9 = \Gamma$	$V(\Gamma_8)$	$E(\Gamma_8) \cup \{\{d_3, d_2\}\}$

6. Artin Groups Based on Cones. If the underlying graph Γ of an Artin group, $A\Gamma$, has the property that there exists $a \in V(\Gamma)$ such that for all $v \in (V\Gamma)$, $\{a, v\} \in E(\Gamma)$ with $k_{\{a,v\}} = 2$, then the kernel of an epimorphism from $A\Gamma$ to $\langle t \rangle$ which sends all generators to t or 1 is particularly tractable. In this section, we will refer to a graph with the above property as a 2–cone, and the vertex a as the apex of Γ. It follows from work in [6] that $\ker \phi$ is finitely presented in the special case that $A\Gamma$ is a graph group. In this section, we give an explicit, finite presentation for the kernels of such epimorphisms when the defining graph of the Artin group is a 2–cone with live apex. In this case, $A\Gamma$ decomposes as $\langle a \rangle \times A\Gamma_1$, where Γ_1 is the base of Γ.

Theorem 3. Let $A\Gamma$ be an Artin group where Γ is a 2–cone with apex a and base Γ_1. If $\phi\colon A\Gamma \to \langle t \rangle$ is an epimorphism mapping all generators to either t or 1 with $\phi(a) = t$, then $\ker \phi \simeq A\Gamma_1$. In particular, $\ker \phi$ is finitely presented.

Proof. For a Schreier transversal of $\ker \phi$ in $A\Gamma$ we choose $T = \{a^n : n \in \mathbb{Z}\}$. As in Section 3, we will denote the family of generators arising from $l_i \in L$ by

$\lambda(i,n)$ and the family of generators arising from $d_i \in D$ by $\delta(i,n)$. The following families of relations arise from each $\{a, l_i\} \in E(\Gamma)$ and $\{a, d_i\} \in E(\Gamma)$, respectively:

$$\lambda(i,n) = \lambda(i,0), \tag{14}$$

$$\delta(i,n) = \delta(i,0). \tag{15}$$

So by Tietze transformation (T4), we can delete these families of relations and, for $n \neq 0$, all the corresponding generators. All other edges in $E(\Gamma)$ correspond to families of relations given in (1)–(5). However, by relations (14) and (15), these families are equivalent to

$$\begin{cases} [\lambda(i,0), \lambda(j,0)]_k = [\lambda(j,0), \lambda(i,0)]_k, \\ [\lambda(i,0), \delta(j,0)]_k = [\lambda(j,0), \delta(i,0)]_k, \\ [\delta(i,0), \delta(j,0)]_k = [\delta(j,0), \delta(i,0)]_k. \end{cases}$$

So by Tietze transformations (T1) and (T2), we can replace each family of relations in (1)–(5) by the corresponding relation above. So each vertex $b \in V(\Gamma_1)$ corresponds to a single generator β in the presentation for $\ker\phi$, and each edge $\{b, c\} \in E(\Gamma_1)$ of weight k corresponds to the single relation $[\beta, \gamma]_k = [\gamma, \beta]_k$ in the presentation of $\ker\phi$. Therefore, $\ker\phi \simeq A\Gamma_1$.

Example 3. Let $\phi\colon A\Gamma \to \langle t \rangle$ be defined by $\phi(a) = \phi(l_i) = t$, and $\phi(d_i) = 1$ where Γ is as follows. (Note: unlabelled edges have arbitrary weight.) Since Γ is a 2–cone, then by Theorem 3, $\ker\phi \simeq A\Gamma_1$ where Γ_1 is shown below.

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