BOUNDED AND L^p–SOLUTIONS TO A GENERALIZED LIENARD EQUATION WITH INTEGRABLE FORCING TERM

Allan Kroopnick

Abstract. This paper presents two theorems concerning the inhomogeneous differential equation $x'' + c(t)f(x)x' + a(t,x) = e(t)$, where $e(t)$ is a continuous absolutely integrable function. The first theorem gives sufficient conditions when all solutions to this equation are bounded while the second discusses when all solutions are in $L^p[0,\infty)$.

In this article we shall discuss, using standard methods, the bounded properties and L^p–properties of solutions of the following generalized Lienard differential equation with forcing term $e(t)$, i.e. the equation,

$$x'' + c(t)f(x)x' + a(t,x) = e(t). \tag{1}$$

Our purpose here is to simplify some previous proofs to equations of this type (see [1] and [2]) as well as extending some of the previous results to include L^p–solutions. Specifically, we shall see under what conditions the solutions to (1) are L^p–solutions. Recall that for a solution x to be an L^p–solution, we must have $\int_0^\infty |x(t)|^p dt < \infty$. L^p–solutions have been previously discussed for the homogeneous case by both Strauss [3] and the author in [4]. Also, see [2] for an excellent summary of previous work concerning this kind of equation as well as for its excellent bibliography. Furthermore, as in [2], we shall not need to make use of Liapunov functions. Finally, the first result will be of such a nature that it covers the case when no damping factor appears, i.e., the case

$$x'' + a(t,x) = e(t). \tag{2}$$

We now state and prove a general boundedness theorem. Without loss of generality, we shall assume $t \geq 0$.

Theorem 1. Given the differential equation in (1), where $e(t)$ is continuous on $[0,\infty)$ and $\int_0^\infty |e(t)|dt < \infty$. Suppose $c(\cdot)$ is continuous on $[0,\infty)$ with $c(\cdot) \geq 0$ and $f(\cdot)$ is continuous on $\mathbb{R}$ with $f(x) \geq 0$. Furthermore, if $a(t,x)$ is continuous on $[0,\infty) \times \mathbb{R}$ with $\int_0^{\pm\infty} a(t,x)dx = \infty$ uniformly in t, $x\frac{\partial}{\partial t}a(t,x) \leq 0$, then any solution $x(\cdot)$ to (1) as well as its derivative $x'(\cdot)$ are bounded on $[0,\infty)$.

Proof. By standard existence theory, there is a solution to (1) which exists on $[0,T)$ for some $T>0$. Multiply equation (1) by x' and perform an integration by parts on the last term on the RHS from 0 to $t<T$ in order to obtain

$$
\begin{aligned}
(4)\qquad & x'(t)^2/2+\int_0^t c(s)f(x(s))x'(s)^2ds+\int_{x(0)}^{x(t)} a(t,u)du \\
& -\int_0^t\int_{x(0)}^{x(s)}\frac{\partial a}{\partial s}(s,u)duds = x'(0)^2/2+\int_0^t e(s)x'(s)ds \\
& \le x'(0)^2/2+\int_0^t |e(s)x'(s)|ds.
\end{aligned}
$$

Now if $x(t)$ is unbounded, then for large values of $|x|$ we have that the LHS of (4) is positive from our hypotheses. By a mean value theorem applied to the second term on the RHS of (4), equation (4) may be rewritten as

$$
\begin{aligned}
(5)\qquad & x'(t)^2/2+\int_0^t c(s)f(x(s))x'(s)^2ds+\int_{x(0)}^{x(t)} a(t,u)du\ ds \\
& -\int_0^t\int_{x(0)}^{x(s)}\frac{\partial a}{\partial s}(s,u(s))duds < x'(0)^2/2+|x'(\bar{t})|K \\
& \left(K=\int_0^\infty |e(t)|dt, 0<\bar{t}<t\right).
\end{aligned}
$$

Now from (5), we see that if $|x|$ approaches ∞, then so must $|x'(t)|$. Otherwise, the LHS of (5) becomes unbounded while the RHS stays bounded which is impossible. Also, as $|x'(t)|$ approaches ∞, so must $|x'(\bar{t})|$ from (5). On any compact subinterval of $[0,T)$, choose t where $x'(t)$ is a maximum. Now integrate equation (1) as before from 0 to t and divide by $x'(t)$ (assume $x'(t) > 0$, a similar argument works for $x'(t) < 0$ except the inequality is reversed) in order to obtain

$$(6) \qquad 1/(x'(t))x'(t)^2/2 + \int_0^t c(s)f(x(s))x'(s)^2 ds + \int_{x(0)}^{x(t)} a(s,u)du$$

$$- \int_0^t \int_{x(0)}^{x(s)} \frac{\partial a}{\partial s}(s,u)duds \leq x'(0)^2/2x'(t) + |x'(\bar{t})|K/x'(t).$$

Now if $x'(t)$ approaches ∞, then the LHS becomes unbounded while the RHS of (6) stays bounded, which is a contradiction. Thus, $|x|$ and $|x'|$ must stay bounded on $[0,T)$. A standard argument now permits the solution to be extended on all of $[0,\infty)$ [5].

Under somewhat stronger conditions, the solutions are L^p–solutions. We now state and prove this theorem.

Theorem 2. The hypotheses are the same as Theorem 1. In addition, suppose $c(t) > c_0 > 0$ for some positive constant c_0, $a(t,x)x \geq a_0 x^p$ $(p > 1)$ for some positive constant a_0, and $c'(t) \leq 0$, then all solutions to (1) are L^p–solutions and x' is square integrable.

Proof. From equation (5) we see that x' must be square integrable since $0 \leq \int_0^t c(t)f(x)x'^2 dt < \infty$ and $\int_0^\infty c(t)f(x)x'^2 dt \geq \int_0^\infty c_0 f_0 x'^2 dt$, where f_0 is a lower bound for $f(x)$ on the interval $[-B,B]$ and B is a bound for x on $[0,\infty)$. In order to see that x is in $L^p[0,\infty)$, we must first multiply equation (1) by x. After integrating from 0 to t and integrating by parts the first term on the LHS, we obtain

$$(7) \qquad x(t)x'(t) - \int_0^t x'(s)^2 ds + \int_0^t c(s)f(x(s))x(s)x'(s)ds$$

$$+ \int_0^t x(s)a(s,x(s))ds = x(0)x'(0) + \int_0^t e(s)x(s)ds.$$

Next, let $F(x) = \int_0^x uf(u)du$. Now upon integration by parts, the above may be rewritten as

$$x(t)x'(t) - \int_0^t x'(s)^2 ds + c(t)F(x(t)) - \int_0^t F(x(s))c'(s)ds + \int_0^t x(s)a(s,x(s))ds \leq K, \tag{8}$$

where $K = |x(0)x'(0)| + \int_0^\infty |e(s)x(s)|ds + |c(0)F(x(0))|$. Since the RHS of (8) is bounded and all terms on the LHS of (7) are either bounded or positive, the result follows.

Remark 1. If we stiffened the requirement that $xa(t,x) \geq 0$ in Theorem 1, the proof would be somewhat simpler since all terms on the LHS of (5) are always positive and we would also have $\int_0^\infty x(t)a(t,x(t))dt < \infty$, using the reasoning of Theorem 2.

Example. Consider the differential equation

$$x'' + k(t)x' + l(t)x^{2m-1} = f(t), \tag{9}$$

where $k(\cdot)$ and $l(\cdot)$ are continuous functions defined for $t \geq 0$, with $k(\cdot)$ and $l(\cdot)$ having continuous non–positive derivatives with $k(t) > k_0 > 0$, $l(t) > l_0 > 0$, and $\int_0^\infty |f(t)|dt < \infty$. Given these conditions, the above theorems show all solutions to (9) are bounded with $\int_0^\infty x'(t)^2 dt < \infty$ and $\int_0^\infty x(t)^{2m} dt < \infty$.

Remark 2. The above results probably cannot be improved much more because of the following example. Consider the equation,

$$x'' + x'/(4(t+1)) + x/(8(t+1)^2) = e(t), \tag{10}$$

where $e(t)$ is continuous and absolutely integrable over $\mathbb{R}$. We then have $x(t) = (t+1)^{1/2}$ is an unbounded solution to the corresponding homogeneous equation (i.e., the case when $e(\cdot) = 0$) so the general solution to (10) is unbounded and yet we have that $c(t) > 0$, $a(t) > 0$, and $f(x) = 1$ after looking at equation (1).

References

1. H. A. Antosiewicz, "On Nonlinear Differential Equations of Second Order with Integrable Forcing Term," *J. Lond. Math. Soc.*, 30 (1955), 64–67.

2. Z. S. Athanasov, "Boundedness Criteria for Solutions of Certain Second Order Nonlinear Differential Equations," *J. Math. Anal. Appl.*, 123 (1987), 461–479.

3. A. Strauss, "Liapunov Functions and L^p–Solutions of Differential Equations," *Trans. Amer. Math. Soc.*, 119 (1965), 37–50.

4. A. J. Kroopnick, "Note on Bounded L^p–Solutions of a Generalized Lienard Equation," *Pacific J. Math.*, 94 (1981), 171–175.

5. J. Hale, *Ordinary Differential Equations*, Interscience, New York, 1969.

Allan Kroopnick
Office of Program Benefits Policy
Social Security Administration
3-D-21 Operations Building
6401 Security Boulevard
Baltimore, MD 21235
email: allan.j.kroopnick@ssa.gov