SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

101. [1997, 34] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Find the coefficient of x^{1997} (in closed form) in the expansion of

$$\sqrt{2x^2 - 3x^3}.$$

Solution by Joseph Wiener, University of Texas–Pan American, Edinburg, Texas.

Write the given radical in the form $\sqrt{2}|x|(1-3x/2)^{1/2}$ and find the coefficient k_{1996} of x^{1996} in the binomial expansion of $(1-3x/2)^{1/2}$, that is,

$$k_{1996} = \binom{1/2}{1996}\left(\frac{3}{2}\right)^{1996},$$

where

$$\binom{1/2}{1996} = \frac{(1/2)!}{1996!(1/2-1996)!} = \frac{\Gamma(3/2)}{1996!\Gamma(-1996+1/2)}.$$

It remains to calculate the values of the gamma function,

$$\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \qquad \Gamma\left(-1996+\frac{1}{2}\right) = \frac{4^{1996}(1996)!}{(2\cdot 1996)!}\sqrt{\pi}.$$

Then

$$k_{1996} = \frac{1}{2}\cdot\frac{3992!}{4^{1996}\cdot(1996!)^2}$$

and the unknown coefficient equals

$$\frac{3992!}{(1996!)^2}\left(\frac{3}{8}\right)^{1996}\frac{\sqrt{2}}{2}\operatorname{sign} x=\binom{3992}{1996}\left(\frac{3}{8}\right)^{1996}\frac{\sqrt{2}}{2}\operatorname{sign} x.$$

Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; N. J. Kuenzi, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; and the proposer.

102. [1997, 34] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

(a) Prove if p is a positive prime of the form $p = 4k + 1$, then k and $-k$ are quadratic residues.

(b) Let p be a positive prime of the form $p = 4k + 3$.

   i. Prove k is a quadratic residue if and only if $-k$ is a quadratic non-residue.

   ii. Prove k is a quadratic residue if $p \equiv 1 \pmod 3$ and k is a quadratic non-residue if $p \equiv 2 \pmod 3$.

Composite solution by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri; and the proposer.

Our solution will use the following results.

($*$) If p is an odd positive prime, then the Legendre symbol

$$\left(-\frac{1}{p}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4\\ -1 & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

Gauss' Quadratic Reciprocity Law

Let p and q be distinct odd primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

For $(*)$, see Corollary on p. 182 of Burton; *Elementary Number Theory*, 3rd edition, Wm. C. Brown Publishers, Dubuque, Iowa, 1994 and for Gauss' Quadratic Reciprocity Law, p. 192 of the same book.

(a) If $p = 4k+1$, then using properties of the Legendre symbol and $(*)$, we have that

$$\left(\frac{k}{p}\right) = \left(\frac{4}{p}\right)\left(\frac{k}{p}\right) = \left(\frac{4k}{p}\right) = \left(\frac{p-1}{p}\right) = \left(\frac{-1}{p}\right) = 1$$

and

$$\left(\frac{-k}{p}\right) = \left(\frac{-1\cdot k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{k}{p}\right) = 1\cdot 1 = 1.$$

Thus k and $-k$ are quadratic residues.

(b) Suppose $p = 4k+3$.

i. If k is a quadratic residue, then

$$\left(\frac{k}{p}\right) = 1.$$

Therefore by $(*)$,

$$\left(\frac{-k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{k}{p}\right) = (-1)(1) = -1.$$

Thus $-k$ is a quadratic non–residue. Similarly, suppose $-k$ is a quadratic non–residue, i.e.,

$$\left(\frac{-k}{p}\right) = -1.$$

Therefore by $(*)$,

$$\left(\frac{k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{-k}{p}\right) = (-1)(-1) = 1.$$

Thus k is a quadratic residue.

ii. By Gauss' Quadratic Reciprocity Law

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}} = -\left(\frac{p}{3}\right),$$

and so

$$\left(\frac{k}{p}\right) = -\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3 \\ -1 & \text{if } p \equiv 2 \pmod 3). \end{cases}$$

Therefore, if $p \equiv 1 \pmod 3$, k is a quadratic residue and if $p \equiv 2 \pmod 3$, k is a quadratic non–residue.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri and Richard Singer, Webster University, St. Louis, Missouri.

103. [1997, 35] *Proposed by Thomas Dence, University of Georgia, Athens, Georgia and Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri.*

Find a closed form for the sum

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \frac{1}{13} - \frac{1}{14} + \cdots .$$

Solution I by Bob Prielipp and John Oman, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; and Donald P. Skow, University of Texas–Pan American, Edinburg, Texas..

Our solution will use the following known results.

(1) If the sequence $\{a_n\}$ is monotonic decreasing with

$$\lim_{n\to\infty} a_n = 0,$$

and the partial sums of $\sum b_n$ are bounded, then $\sum a_n b_n$ converges.

(2) If $\sum a_n$ converges to s, every series $\sum b_n$ obtained from $\sum a_n$ by inserting parentheses also converges to s.
(3) (Abel's Limit Theorem) Assume that we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad -r < x < r.$$

If the series also converges at $x = r$, then the limit

$$\lim_{x \to r^-} f(x)$$

exists and

$$\lim_{x \to r^-} f(x) = \sum_{n=0}^{\infty} a_n r^n.$$

For (1), see Corollary 1 on p. 113 of Buck, *Advanced Calculus*, McGraw–Hill Book Company, Inc, New York, 1956; for (2), see 12–13 Theorem on p. 357 of Apostol, *Mathematical Analysis: A Modern Approach to Advanced Calculus*, Addison–Wesley Publishing Company, Inc., Reading, Massachusetts, 1957; and for (3), see 13-33 Theorem on p. 421 of Apostol's *Mathematical Analysis* given above.

Letting $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/4$, $a_4 = 1/5$, $a_5 = 1/7$, etc. and

$$\sum b_n = \sum_{n=1}^{\infty} (-1)^{n+1},$$

it follows from (1) that the given series converges (note that the partial sums of $\sum b_n$ are always either 1 or 0). From (2), the given series has the same sum as the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} - \frac{1}{3n-1} \right).$$

(The latter series converges (absolutely) by the integral test.)

$$\int_1^\infty \left(\frac{1}{3x-2} - \frac{1}{3x-1}\right) dx = \lim_{N\to\infty} \int_1^N \left(\frac{1/3}{3x-2} - \frac{1/3}{3x-1}\right)(3dx)$$

$$= \lim_{N\to\infty} \left(\frac{1}{3}\ln(3x-2) - \frac{1}{3}\ln(3x-1)\right)\Bigg|_1^N = \frac{1}{3}\lim_{N\to\infty}\left(\ln\left(\frac{3x-2}{3x-1}\right)\Bigg|_1^N\right)$$

$$= \frac{1}{3}\lim_{N\to\infty}\left(\ln\left(\frac{3N-2}{3N-1}\right) - \ln\frac{1}{2}\right) = \frac{1}{3}(0-(-\ln 2)) = \frac{1}{3}\ln 2.$$

Set

$$f(x) = x - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots + \frac{x^{3n-2}}{3n-2} - \frac{x^{3n-1}}{3n-1} + \cdots,$$

where $0 < x \le 1$. The series is absolutely convergent for $|x| < 1$, and therefore we can rearrange the terms to get

$$f(x) = \left(x + \frac{x^4}{4} + \frac{x^7}{7} + \cdots + \frac{x^{3n-2}}{3n-2} + \cdots\right) - \left(\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{8} + \cdots + \frac{x^{3n-1}}{3n-1} + \cdots\right).$$

Now for $0 < x < 1$,

$$f'(x) = (1 + x^3 + x^6 + \cdots + x^{3n-3} + \cdots) - (x + x^4 + x^7 + \cdots + x^{3n-2} + \cdots)$$

$$= \frac{1}{1-x^3} - \frac{x}{1-x^3} = \frac{1-x}{1-x^3} = \frac{1}{1+x+x^2}.$$

But,

$$\int \frac{1}{1+x+x^2}dx = \int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(x+\frac{1}{2}\right)^2}dx$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

Because $f(0) = 0$,

$$f(x) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}.$$

Since the series representation of f is convergent for $x = 1$ by the alternating series test, (3) implies that the original series converges to

$$f(1) = \frac{2}{\sqrt{3}} \tan^{-1}(\sqrt{3}) - \frac{\pi}{3\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3} - \frac{\pi}{3\sqrt{3}} = \frac{\pi}{3\sqrt{3}} = 0.6045997881.$$

Solution II by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

Our solution will use the following known results.

(1) If the sequence $\{a_n\}$ is monotonic decreasing with

$$\lim_{n\to\infty} a_n = 0,$$

and the partial sums of $\sum b_n$ are bounded, then $\sum a_n b_n$ converges.

(2) If $\sum a_n$ converges to s, every series $\sum b_n$ obtained from $\sum a_n$ by inserting parentheses also converges to s.

For (1), see Corollary 1 on p. 113 of Buck, *Advanced Calculus*, McGraw–Hill Book Company, Inc, New York, 1956 and for (2), see 12–13 Theorem on p. 357 of Apostol, *Mathematical Analysis: A Modern Approach to Advanced Calculus*, Addison–Wesley Publishing Company, Inc., Reading, Massachusetts, 1957.

Letting $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/4$, $a_4 = 1/5$, $a_5 = 1/7$, etc. and

$$\sum b_n = \sum_{n=1}^{\infty} (-1)^{n+1},$$

it follows from (1) that the given series converges (note that the partial sums of $\sum b_n$ are always either 1 or 0). From (2), the given series has the same sum as the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}-\frac{1}{3n-1}\right).$$

(The latter series converges (absolutely) by the integral test.)

$$\int_1^{\infty}\left(\frac{1}{3x-2}-\frac{1}{3x-1}\right)dx = \lim_{N\to\infty}\int_1^{N}\left(\frac{1/3}{3x-2}-\frac{1/3}{3x-1}\right)(3dx)$$

$$= \lim_{N\to\infty}\left(\frac{1}{3}\ln(3x-2)-\frac{1}{3}\ln(3x-1)\right)\Big|_1^N = \frac{1}{3}\lim_{N\to\infty}\left(\ln\left(\frac{3x-2}{3x-1}\right)\Big|_1^N\right)$$

$$= \frac{1}{3}\lim_{N\to\infty}\left(\ln\left(\frac{3N-2}{3N-1}\right)-\ln\frac{1}{2}\right) = \frac{1}{3}(0-(-\ln 2)) = \frac{1}{3}\ln 2.$$

Set

$$f(x) = x - \frac{x^2}{2}+\frac{x^4}{4}-\frac{x^5}{5}+\cdots+\frac{x^{3n-2}}{3n-2}-\frac{x^{3n-1}}{3n-1}+\cdots,$$

where $0 < x \leq 1$. The series is absolutely convergent for $|x| < 1$, and therefore we can rearrange the terms.

It is known that

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} = \begin{cases} \frac{\pi}{4} & \text{if } 0 < x < \pi \\ 0 & \text{if } x = 0 \text{ or } x = \pi \\ -\frac{\pi}{4} & \text{if } \pi < x < 2\pi. \end{cases}$$

Letting $x = \pi/3$ yields

$$1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\cdots = \frac{\pi}{2\sqrt{3}}.$$

See p. 386 of Friedman, *Advanced Calculus*, Holt, Rinehart and Winston, Inc., New York, 1971. Thus

$$\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\frac{1}{13}-\frac{1}{14}+\frac{1}{16}-\frac{1}{17}+\frac{1}{19}-\frac{1}{20}+\cdots\right)=$$

$$\left(1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\cdots\right)-\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{10}+\frac{1}{14}-\frac{1}{16}+\cdots\right)$$

$$=\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\cdots\right).$$

Hence, if the sum of the given infinite series is S, then

$$S=\frac{\pi}{2\sqrt{3}}-\frac{1}{2}S$$

so

$$S=\frac{\pi}{3\sqrt{3}}.$$

Solution III by N. J. Kuenzi, John Oman, and Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

Our solution is based on two properties and two specific values of the psi function. The psi function is defined by

$$\mathrm{psi}(x)=\frac{\mathrm{d}}{\mathrm{dx}}\ln(\Gamma(x)).$$

The properties and values used are

$$(i)\qquad \mathrm{psi}(x+n)=\mathrm{psi}(x)+\sum_{k=0}^{n-1}\frac{1}{x+k},$$

$$(ii)\qquad \lim_{n\to\infty}[\mathrm{psi}(z+n)-\ln(n)]=0,$$

$$(iii) \qquad \operatorname{psi}\left(\frac{1}{3}\right) = \operatorname{psi}(1) - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3, \text{ and}$$

$$(iv) \qquad \operatorname{psi}\left(\frac{2}{3}\right) = \operatorname{psi}(1) + \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3.$$

(*Table of Integrals, Series, and Products*, Gradshteyu and Ryzhik, Academic Press, 1980, pp. 943–946.)

The given series is an alternating series with monotone decreasing terms and hence converges, so we can write the series in the form

$$\sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+2}\right).$$

Let

$$S(n) = \sum_{k=0}^{n}\left(\frac{1}{3k+1} - \frac{1}{3k+2}\right) = \frac{1}{3}\sum_{k=0}^{n}\frac{1}{k+\frac{1}{3}} - \frac{1}{3}\sum_{k=0}^{n}\frac{1}{k+\frac{2}{3}}.$$

Using (i) yields,

$$S(n) = \frac{1}{3}\left[\operatorname{psi}\left(n+\frac{4}{3}\right) - \operatorname{psi}\left(\frac{1}{3}\right)\right] - \frac{1}{3}\left[\operatorname{psi}\left(n+\frac{5}{3}\right) - \operatorname{psi}\left(\frac{2}{3}\right)\right]$$

$$= \frac{1}{3}\left[\left(\operatorname{psi}\left(n+\frac{4}{3}\right) - \ln(n)\right) - \operatorname{psi}\left(n+\frac{5}{3}\right) + \ln(n)\right] + \frac{1}{3}\left[\operatorname{psi}\left(\frac{2}{3}\right) - \operatorname{psi}\left(\frac{1}{3}\right)\right].$$

So,

$$S(n) = \frac{1}{3}\left[\left(\operatorname{psi}\left(n+\frac{4}{3}\right) - \ln(n)\right) - \left(\operatorname{psi}\left(n+\frac{5}{3}\right) - \ln(n)\right)\right] + \frac{\pi\sqrt{3}}{9}.$$

It follows from (ii) that

$$S(n) \to \frac{\pi\sqrt{3}}{9} \text{ as } n \to \infty.$$

So,

$$\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}-\frac{1}{3k+2}\right)=\frac{\pi\sqrt{3}}{9}.$$

Solution IV by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

We shall use the result established by Cerimele in "Summation of Generalized Harmonic Series with Periodic Sign Distributions." This article appears on pp. 331–335 of the Spring 1968 issue of the *Pi Mu Epsilon Journal.* Since

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\frac{1}{13}-\frac{1}{14}+\cdots=$$

$$\frac{1}{1}+\frac{-1}{2}+\frac{1}{4}+\frac{-1}{5}+\frac{1}{7}+\frac{-1}{8}+\frac{1}{10}+\frac{-1}{11}+\frac{1}{13}+\frac{-1}{14}+\cdots=$$

$$\frac{(-1)^0}{1+\lfloor\frac{0}{2}\rfloor 3+0\cdot 1}+\frac{(-1)^1}{1+\lfloor\frac{1}{2}\rfloor 3+1\cdot 1}+\frac{(-1)^2}{1+\lfloor\frac{2}{2}\rfloor 3+0\cdot 1}$$

$$+\frac{(-1)^3}{1+\lfloor\frac{3}{2}\rfloor 3+1\cdot 1}+\frac{(-1)^4}{1+\lfloor\frac{4}{2}\rfloor 3+0\cdot 1}+\cdots=$$

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{1+\lfloor\frac{i}{2}\rfloor 3+(i \bmod 2)\cdot 1}$$

(where $\lfloor x\rfloor$ is the greatest integer in x), the sum we are seeking in Cerimele's notation is $W(1;1,2)$ which Cerimele shows is equal to

$$\frac{1}{3}(\mathrm{psi}(2/3)-\mathrm{psi}(1/3))$$

$$=\frac{1}{3}\left(\left(-\gamma-\frac{3}{2}\ln 3+\frac{\pi}{2\sqrt{3}}\right)-\left(-\gamma-\frac{3}{2}\ln 3-\frac{\pi}{2\sqrt{3}}\right)\right)$$

$$=\frac{\pi}{3\sqrt{3}}=\frac{1}{9}(\pi\sqrt{3}).$$

Here, γ denotes Euler's constant. Also, the last set of equalities follow from p. 332 and p. 334 of the article cited above.

Comment by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico. A similar result can be found in L. C. Larson's *Problem–Solving Through Problems* (Problem 6.9.2, p. 235) published by Springer–Verlag, New York, 1983.

Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri and the proposers. One incorrect solution was also received.

104. [1997, 35] *Proposed by Kenneth Davenport, P. O. Box 99901, Pittsburgh, Pennsylvania.*

Show that

$$1 \cdot \sin\frac{\pi}{2n} + 3 \cdot \sin\frac{3\pi}{2n} + 5 \cdot \sin\frac{5\pi}{2n} + \cdots + (2n-1)\sin\frac{(2n-1)\pi}{2n} = n\csc\frac{\pi}{2n}.$$

Solution I by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico and Joseph Wiener, University of Texas–Pan American, Edinburg, Texas. We use

$$\sum_{j=1}^{n}\cos(2j-1)x = \frac{\sin 2nx}{2\sin x}.$$

(Note: To prove this, multiply through by $2\sin x$ and use the formula

$$2\cos u\sin v = \sin(u+v) - \sin(u-v).$$

The result is $\sin 2nx$.)

Now differentiating

$$\sum_{j=1}^{n}\bigl(-\sin(2j-1)x\bigr)(2j-1) = \frac{\sin x\cos 2nx \cdot 2n - \sin 2nx(\cos x)}{2\sin^2 x}.$$

Let $x = \pi/(2n)$. Then

$$\sum_{j=1}^{n}(2j-1)\sin(2j-1)\frac{\pi}{2n} = \frac{2n\sin\frac{\pi}{2n}}{2\sin^2\frac{\pi}{2n}} = n\csc\frac{\pi}{2n}.$$

This is the desired result.

Solution II by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri and Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri.

Applying the operator $r\frac{d}{dr}$ to the identity

$$\sum_{j=0}^{n-1} r^{2j+1} = \frac{r - r^{2n+1}}{1 - r^2}$$

for $r \neq \pm 1$ gives

$$(1) \qquad \sum_{j=0}^{n-1} (2j+1) r^{2j+1} = \frac{(2n-1)r^{2n+3} - (2n+1)r^{2n+1} + r^3 + r}{(1-r^2)^2}.$$

Substituting $r = e^{\pi i/2n}$ into (1) and simplifying yields

$$(2) \qquad \sum_{j=0}^{n-1} (2j+1) e^{(2j+1)\pi i/2n} = \frac{(n-1)e^{\frac{\pi i}{2n}} - (n+1)e^{-\frac{\pi i}{2n}}}{2\sin^2 \frac{\pi}{2n}}.$$

The desired result follows by taking the imaginary parts of (2) and simplifying.

Euler and Sadek noted that taking the real parts of identity (2) and simplifying gives

$$\sum_{j=0}^{n-1} (2j+1) \cos \frac{(2j+1)\pi}{2n} = -\cot \frac{\pi}{2n} \csc \frac{\pi}{2n}.$$

They also noted that, in general, if we let $r = e^{i\theta}$ in (1), the denominator becomes

$$(1-r^2)^2 = (1-e^{2i\theta})^2 = (e^{i\theta}(e^{-i\theta} - e^{i\theta}))^2$$

$$= e^{2i\theta}(-2i\sin\theta)^2 = e^{2i\theta}(-4\sin^2\theta),$$

and the right hand side of (1) becomes

$$\begin{aligned}
&\frac{e^{3i\theta} + (2n-1)e^{(2n+3)i\theta} - (2n+1)e^{(2n+1)i\theta} + e^{i\theta}}{-4e^{2i\theta}\sin^2\theta} \\
&= \frac{e^{i\theta} + (2n-1)e^{(2n+1)i\theta} - (2n+1)e^{(2n-1)i\theta} + e^{-i\theta}}{-4\sin^2\theta} \\
&= \frac{2\cos\theta + (2n-1)e^{(2n+1)i\theta} - (2n+1)e^{(2n-1)i\theta}}{-4\sin^2\theta}.
\end{aligned}$$

Taking the real and imaginary parts yields the following two identities.

$$\begin{aligned}
&\sum_{j=0}^{n-1}(2j+1)\cos((2j+1)\theta) \\
&\qquad = \frac{2\cos\theta + (2n-1)\cos((2n+1)\theta) - (2n+1)\cos((2n-1)\theta)}{-4\sin^2\theta}
\end{aligned}$$

$$\sum_{j=0}^{n-1}(2j+1)\sin((2j+1)\theta) = \frac{(2n-1)\sin((2n+1)\theta) - (2n+1)\sin((2n-1)\theta)}{-4\sin^2\theta}.$$

Also solved by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin (2 solutions) and the proposer.