FINITE PRESENTATIONS OF SUBGROUPS OF GRAPH GROUPS

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Abstract. Given a finite simple graph Γ, the graph group $G\Gamma$ is the group with generators in one–to–one correspondence with the vertices of Γ and with relations in one–to–one correspondence with the edges of Γ: two generators commute if and only if their associated vertices are adjacent in Γ. Let $\phi\colon G\Gamma \to \mathbb{Z}$ be a homomorphism which maps each generator to 0 or 1. We derive an explicit presentation for $\ker\phi$, and give a condition, dependent on Γ and ϕ, which guarantees the finite presentation of $\ker\phi$.

1. Introduction. Free partially commutative (FPC) monoids were first introduced by P. Cartier and D. Foata [2] in order to study combinatorial problems involving rearrangements of words. The corresponding FPC groups are known as graph groups.

Various properties of graph groups have already been determined. The word and conjugacy problems for groups were shown to be solvable by C. Wrathall [16, 17] and independently by H. Servatius [11]. Graph groups were shown to satisfy quadratic isoperimetric and linear isodiametric inequalities by J. Meier [8]. They were shown to be biautomatic by the third author [14] and independently by S. Hermiller and J. Meier [5] using the more general notion of a "graph product" of groups. C. Droms [3, 4, 12], H. Servatius [12], B. Servatius [12] and J. Meier and the third author [9] have studied various subgroups of graph groups.

The focus of this paper is the kernels of homomorphisms from graph groups to the infinite cyclic group $\langle t\rangle$ in which each generator is mapped to t or 1. In [9], necessary and sufficient conditions were given for a generalization of such kernels to be finitely generated. Here, the Reidemeister–Schreier method is used to find presentations for these kernels and a sufficient condition for their finite presentation is given. Although not apparent at first, this condition on the constructibility of Γ will force the "living subgraph" to be connected and a dominating subgraph of Γ, as in [9].

The main result of this paper, the corollary to Theorem 1, is a condition based on the constructibility of the graph Γ–partitioned according to the mapping of the vertices under the homomorphism — from a single vertex using two types of operations to add edges and vertices. This constructive condition can be phrased in terms of simple homotopic expansions of the simplicial complexes spanned by the representative graphs of the graph groups, as we show in the last section.

Bestvina and Brady [1] have established the existence of groups of homological type FP_2 which are not finitely presented. The group presentations derived here are applicable to such groups.

2. Preliminaries. Given a graph Γ, we will denote the set of vertices of Γ by $V(\Gamma)$ and the set of edges, consisting of unordered pairs of elements of $V(\Gamma)$, by $E(\Gamma)$. A simple graph is a graph with no loops or multiple edges.

A finite simple graph Γ induces a presentation of a group $G\Gamma$:

$$\langle V(\Gamma); xy = yx \text{ for all } \{x, y\} \in E(\Gamma)\rangle.$$

A group G is called a graph group provided there exists some finite simple graph Γ such that $G \simeq G\Gamma$. Given any graph group, we will assume the above presentation and identify the set of generators of $G\Gamma$ with the vertex set $V(\Gamma)$.

Following [7], we will denote the Tietze transformations adjunction of relators, deletion of relators, adjunction of generators, and deletion of generators by (T1), (T2), (T3), and (T4), respectively. It is shown in [7] that given any two presentations of a group G, one can be obtained from the other by repeated applications of these four transformations.

Given a presentation of a group G and suitable information about a subgroup $H \leq G$, the Reidemeister–Schreier method enables one to obtain a presentation for H. In [6], it is shown that if $\langle X; R\rangle$ is a presentation for G, $\pi\colon F(X) \to G$ is the canonical epimorphism, T is a Schreier transversal for $\pi^{-1}(H)$ in $F(X)$, and $\Phi\colon F(X) \to T$ is the function which maps each element to its coset representative, then $Y = \{tx\Phi(tx)^{-1} : t \in T, x \in X, tx \notin T\}$ corresponds to a set of generators for H. Furthermore, if $\tau\colon \pi^{-1}(H) \to F(Y)$ is the function in [6] which rewrites each $w \in \pi^{-1}(H)$ in terms of the generators Y and $S = \{\tau(trt^{-1}) : t \in T, r \in R\}$, then $\langle Y; S\rangle$ is a presentation for H.

The reader is directed to [6] or [7] for details of Tietze transformations and the Reidemeister–Schreier rewriting procedure.

3. The Subgroups. Let Γ be a finite simple graph and $G\Gamma$ its corresponding graph group. Partition $V(\Gamma)$ into two sets, $A = \{a_0, a_1, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_s\}$ and define the homomorphism $\phi\colon G\Gamma \to \langle t\rangle$ by $\phi(a_i) = t$ and $\phi(b_i) = 1$ for all indices i. Throughout this paper, we will assume $A \neq \emptyset$. Following [9], we will refer to the vertices not mapped to 1 — in this case, the elements of A —

as live, and denote the full subgraph spanned by these vertices as $\mathcal{L} = \mathcal{L}(\phi)$, the living subgraph of Γ. For a Schreier transversal for $\ker\phi$ in $G\Gamma$, take $T = \{a_0^n : n \in \mathbb{Z}\}$ and recall that $\Phi\colon G\Gamma \to T$ sends each element to its coset representative. A set of generators of $\ker\phi$ is then $\{tx\Phi(tx)^{-1} : t \in T, x \in X, tx \notin T\}$, which we denote by

$$\alpha(i,n) = a_0^n a_i a_0^{-(n+1)},$$
$$\beta(j,n) = a_0^n b_j a_0^{-n}$$

for $i = 1,2,\ldots,r$, $j = 1,2,\ldots,s$, and $n \in \mathbb{Z}$. We now obtain the defining set of relations for $\ker\phi$ by rewriting the set $\{trt^{-1} : t \in T, r \in R\}$, where $R = \{xyx^{-1}y^{-1} : \{x,y\} \in E(\Gamma)\}$, in terms of these generators. For convenience, we define $\alpha(0,n) = 1$ for all $n \in \mathbb{Z}$. Straightforward computations show that each $\{a_i,a_j\} \in E(\Gamma)$ yields the family of relations

$$\alpha(i,n)\alpha(j,n+1)\alpha(i,n+1)^{-1}\alpha(j,n)^{-1} = 1, \quad n \in \mathbb{Z},$$

while each $\{a_i,b_j\} \in E(\Gamma)$ yields the family

$$\alpha(i,n)\beta(j,n+1)\alpha(i,n)^{-1}\beta(j,n)^{-1} = 1, \quad n \in \mathbb{Z},$$

and each $\{b_i,b_j\} \in E(\Gamma)$ yields the family

$$\beta(i,n)\beta(j,n)\beta(i,n)^{-1}\beta(j,n)^{-1} = 1, \quad n \in \mathbb{Z}.$$

Thus, $\ker\phi$ can be presented using the above generators together with the relations

$$\begin{cases} \alpha(i,n)\alpha(j,n+1) = \alpha(j,n)\alpha(i,n+1), & \text{for } \{a_i,a_j\} \in E(\Gamma),\ n \in \mathbb{Z}, \\ \alpha(i,n)\beta(j,n+1) = \beta(j,n)\alpha(i,n), & \text{for } \{a_i,b_j\} \in E(\Gamma),\ n \in \mathbb{Z}, \\ \beta(i,n)\beta(j,n) = \beta(j,n)\beta(i,n), & \text{for } \{b_i,b_j\} \in E(\Gamma),\ n \in \mathbb{Z}. \end{cases} \tag{1}$$

Note that every element of $V(\Gamma) - \{a_0\}$ corresponds to a countable family of generators and every element of $E(\Gamma)$ corresponds to a countable family of relations in this presentation for $\ker\phi$.

4. Some Lemmas. We are now ready to prove the following lemmas, which will aid us in our goal of finding sufficient conditions for the finite presentation of $\ker\phi$.

Lemma 1. Let Γ be a finite simple graph and let $V(\Gamma)$ be partitioned into two sets A and B. Define the homomorphism $\phi: G\Gamma \to \langle t\rangle$ so that $\phi(x) = t$ for every $x \in A$ and $\phi(x) = 1$ for every $x \in B$. Let $\hat{\Gamma}$ be the graph constructed from Γ by adding a vertex c and an edge $\{c, a\}$ for some $a \in A$. If $\hat{\phi}: G\hat{\Gamma} \to \langle t\rangle$ is a homomorphism that agrees with ϕ on $V(\Gamma)$ and maps c to either t or 1, then there exist presentations for $\ker\phi$ and $\ker\hat{\phi}$ of the form $\langle X; R\rangle$ and $\langle X \cup \{\gamma\}; R\rangle$, respectively, for some generating symbol γ independent of X.

Proof. We will use the presentation for $\ker\phi$ calculated in the previous section and denote it by $\langle X; R\rangle$. The vertices of $\hat{\Gamma}$ may be partitioned into two subsets, $\hat{A}$ and $\hat{B}$, such that $\hat{\phi}$ maps every element of $\hat{A}$ to t and every element of $\hat{B}$ to 1. Since exactly one vertex and one edge have been added to Γ, $\ker\hat{\phi}$ may be presented using the generators X and relators R together with one additional family of generators and one additional family of relations. We denote the family of generators corresponding to the vertex a by α_n and the family of generators corresponding to c by γ_n for $n \in \mathbb{Z}$.

We first consider the case where $c \in \hat{A}$. With our new notation, the first family in (1) becomes

$$\alpha_n\gamma_{n+1} = \gamma_n\alpha_{n+1}, \quad n \in \mathbb{Z}. \tag{2}$$

Straightforward induction shows (2) is equivalent to

$$\gamma_n = \begin{cases} [\alpha_0\alpha_1\cdots\alpha_{n-1}]^{-1}\gamma_0[\alpha_1\alpha_2\cdots\alpha_n], & n > 0, \\ [\alpha_n\alpha_{n+1}\cdots\alpha_{-1}]\gamma_0[\alpha_{n+1}\alpha_{n+2}\cdots\alpha_0]^{-1}, & n < 0. \end{cases} \tag{3}$$

By the Tietze transformations (T1) and (T2), the family of relations (2) can be replaced by the family (3). Since for all nonzero integers n, γ_n has been written in terms of γ_0 and elements of X, by (T4) we can eliminate every relation in the family (3) and every generator γ_n for $n \neq 0$. The generators γ_n for $n \neq 0$ do not appear in any of the relations in R, so $\langle X \cup \{\gamma_0\}; R\rangle$ is a presentation for $\ker\hat{\phi}$.

Next, we consider the case where $c \in \hat{B}$. The second family of relations in (1) gives

$$\alpha_n\gamma_{n+1} = \gamma_n\alpha_n, \quad n \in \mathbb{Z}. \tag{4}$$

Proceeding as in the first case, we replace (4) by

$$\gamma_n = \begin{cases} [\alpha_0\alpha_1\cdots\alpha_{n-1}]^{-1}\gamma_0[\alpha_0\alpha_1\cdots\alpha_{n-1}], & n > 0, \\ [\alpha_n\alpha_{n+1}\cdots\alpha_{-1}]\gamma_0[\alpha_n\alpha_{n+1}\cdots\alpha_{-1}]^{-1}, & n < 0. \end{cases} \tag{5}$$

and eliminate every relation in (5) and all the generators γ_n for $n \neq 0$. So we again see that $\langle X \cup \{\gamma_0\}; R\rangle$ is a presentation for $\ker \hat{\phi}$, concluding our proof.

Lemma 2. Let Γ be a finite simple graph and let $V(\Gamma)$ be partitioned into two sets A and B. Define the homomorphism $\phi: G\Gamma \to \langle t\rangle$ so that $\phi(x) = t$ for every $x \in A$ and $\phi(x) = 1$ for every $x \in B$. Let $\hat{\Gamma}$ be the graph constructed from Γ by adding an edge that joins two vertices that are both adjacent to the same vertex $a \in A$. If $\hat{\phi}: G\hat{\Gamma} \to \langle t\rangle$ is the homomorphism that agrees with ϕ on $V(\Gamma)$, then there exists presentations for $\ker \phi$ and $\ker \hat{\phi}$ of the form $\langle X; R\rangle$ and $\langle X; R \cup \{r\}\rangle$, respectively, for some relation r not derivable from R.

Proof. We will use the same presentation $\langle X; R\rangle$ for $\ker \phi$ as in Lemma 1. There are three possible cases for the two vertices joined by the new edge: either both are elements of A, one is in A and one is in B, or both are elements of B.

For the first case, suppose these two vertices, say a' and a'', are both contained in A. Using our established generators and relations, we see the vertices a, a', and a'' are associated with three families of generators, which we will call α_n, α'_n, and α''_n, respectively, where $n \in \mathbb{Z}$, and that the edges $\{a, a'\}$ and $\{a, a''\}$ correspond to the two sets of relations contained in R,

$$\alpha_n \alpha'_{n+1} = \alpha'_n \alpha_{n+1}, \quad n \in \mathbb{Z}, \tag{6}$$
$$\alpha_n \alpha''_{n+1} = \alpha''_n \alpha_{n+1}, \quad n \in \mathbb{Z}. \tag{7}$$

The added edge $\{a', a''\}$ introduces the new relations

$$\alpha'_n \alpha''_{n+1} = \alpha''_n \alpha'_{n+1}, \quad n \in \mathbb{Z}. \tag{8}$$

However, we can use only the relation for $n = 0$,

$$\alpha'_0 \alpha''_1 = \alpha''_0 \alpha'_1, \tag{9}$$

together with the families (6) and (7) to derive the entire family (8) as follows. We start with (9) and induct on n in both the positive and the negative directions. Suppose

$$\alpha'_k \alpha''_{k+1} = \alpha''_k \alpha'_{k+1} \tag{10}$$

for a particular $k \in \mathbb{Z}$. Rewriting every symbol in (10) using the two relations

$$\alpha'_k = \alpha_k \alpha'_{k+1} \alpha_{k+1}^{-1},$$
$$\alpha''_k = \alpha_k \alpha''_{k+1} \alpha_{k+1}^{-1},$$

from (6) and (7) yields

$$\alpha'_{k+1}\alpha''_{k+2} = \alpha''_{k+1}\alpha'_{k+2},$$

completing our induction in the positive direction. For the negative direction, we again use two substitutions derived from (6) and (7) to rewrite (10) as

$$\alpha'_{k-1}\alpha''_{k} = \alpha''_{k-1}\alpha'_{k}.$$

We have now shown that (8) can be derived from (9) and relations in R. Hence, (T2) can be used to delete all relations in (8) except the single relation (9), allowing $\ker\hat{\phi}$ to be presented by $\langle X; R \cup \{\alpha'_0\alpha''_1 = \alpha''_0\alpha'_1\}\rangle$, as promised.

For the second case, we suppose one vertex a' is contained in A and one vertex b' is in B and denote their corresponding generators by α'_n and β'_n for $n \in \mathbb{Z}$. The added edge $\{a', b'\}$ introduces the relations

$$\alpha'_n\beta'_{n+1} = \beta'_n\alpha'_n, \quad n \in \mathbb{Z}. \tag{11}$$

Again, the relation for $n = 0$,

$$\alpha'_0\beta'_1 = \beta'_0\alpha'_0, \tag{12}$$

and the relations of the edges $\{a, b'\}$ and $\{a, a'\}$ can be used to derive each of the relations of (11). The same induction technique is used, with one pair of substitutions to increment the indices on the generators and one pair to decrement them. Hence, (T2) can be used to delete all the new relations except for (12), giving the presentation $\langle X; R \cup \{\alpha'_0\beta'_1 = \beta'_0\alpha'_0\}\rangle$.

Finally, we consider the case where both vertices b' and b'' are elements of B. Once again, we wish to eliminate the family of relations in the sets of generators β'_n and β''_n introduced by the new edge $\{b', b''\}$,

$$\beta'_n\beta''_n = \beta''_n\beta'_n, \quad n \in \mathbb{Z}. \tag{13}$$

The special case $\beta'_0\beta''_0 = \beta''_0\beta'_0$, together with the families of relations corresponding to the edges $\{a, b'\}$ and $\{a, b''\}$ yield (13) by an induction similar to the previous two cases, so a single relation can take the place of the infinite set, giving $\ker\hat{\phi}$ the presentation $\langle X; R \cup \{\beta'_0\beta''_0 = \beta''_0\beta'_0\}\rangle$.

Thus, we have shown that in each case there exists a relation r such that $\ker\hat{\phi} = \langle X; R \cup \{r\}\rangle$. We also know that r is not derivable from R, for if it were, we would have $\ker\phi = \ker\hat{\phi}$. But the word $xyx^{-1}y^{-1}$ represents the trivial element in $\ker\hat{\phi}$ but not in $\ker\phi$, where $x, y \in V(\Gamma)$ are the two vertices joined by the new edge.

Example. Consider the three graphs

where $\phi: G\Gamma \to \langle t \rangle$ satisfies $\phi(a) = \phi(a') = \phi(a'') = t$ and $\phi(b) = 1$. Let ϕ_1 and ϕ_2 be the homomorphisms corresponding to Γ_1 and Γ_2, respectively, with the same vertex mappings as ϕ. Repeated use of Lemma 1 shows $\ker \phi$ is the free group on three letters. However, $\ker \phi_1$ is not finitely presented [13]. On the other hand, since Γ_2 can be constructed from Γ by two applications of Lemma 2 (first adding the edge $\{a, a''\}$, and then adding the edge $\{a'', b\}$), it follows from Lemma 2 that $\ker \phi_2$ admits a presentation with three generators and two relations.

Lemma 3. Let G, G_1, and G_2 be groups with presentations $\langle X; R \rangle$, $\langle X \cup \{\gamma\}; R \rangle$, and $\langle X; R \cup \{r\} \rangle$, respectively, for a generating symbol γ independent of X and a relation r not derivable from R. If G admits a second presentation $\langle X'; R' \rangle$, then $G_1 \simeq \langle X' \cup \{\gamma'\}; R' \rangle$ and $G_2 \simeq \langle X'; R' \cup \{r'\} \rangle$, where γ' is a generating symbol independent of X' and r' is not derivable from R'.

Proof. Since none of the relations of R involve γ,

$$G_1 \simeq \langle X \cup \{\gamma\}; R \rangle \simeq \langle X; R \rangle * \langle \gamma \rangle \simeq \langle X'; R' \rangle * \langle \gamma \rangle \simeq \langle X' \cup \{\gamma'\}; R' \rangle.$$

For the third group, $G_2 \simeq G/N$, where N is the normal subgroup of G generated by the group element g corresponding to r [7]. Thus, if $\pi: F(X) \to G_2$ and $\pi': F(X') \to G_2$ are the canonical epimorphisms, $G_2 \simeq G/N \simeq \langle X'; R' \cup \{r'\} \rangle$, where r' satisfies $g = \pi(r) = \pi'(r')$. Since r is not derivable from R by hypothesis, r' is not derivable from R'.

5. A Condition for Finite Presentation. The first two lemmas provide a method for adding vertices and edges to our graph while allowing us to keep track of the generators and relations of the kernel. Specifically, we have found two operations which may be used to modify the graph and homomorphism so that either one generator or one relation is added to a certain presentation for the kernel. With our first theorem, we use Lemma 3 to generalize this method, applying it to any presentation of the kernel.

Theorem 1. Let Γ be a finite simple graph and let $V(\Gamma)$ be partitioned into two sets A and B. Define the homomorphism $\phi: G\Gamma \to \langle t \rangle$ so $\phi(x) = t$ for every $x \in A$ and $\phi(x) = 1$ for every $x \in B$. Construct a graph $\hat{\Gamma}$ from Γ by either

(i) adding a vertex c and an edge $\{c, a\}$, where $a \in A$; or

(ii) adding an edge joining two vertices which are both adjacent to the same vertex $a \in A$,

and let $\hat{\phi}: G\hat{\Gamma} \to \langle t \rangle$ be a homomorphism that maps every element of $V(\hat{\Gamma})$ to either t or 1 and agrees with ϕ on $V(\Gamma)$. Suppose $\langle X'; R' \rangle$ is any presentation for $\ker \phi$. Then in the first case, $\ker \hat{\phi}$ admits the presentation $\langle X' \cup \{\gamma\}; R' \rangle$, for some generating symbol γ independent of X', and in the second case, $\ker \hat{\phi}$ admits the presentation $\langle X'; R' \cup \{r'\} \rangle$, where r' is a relation not derivable from R'.

Proof. In construction (i) of $\hat{\Gamma}$, Lemma 1 shows there exists a presentation $\langle X \cup \{\gamma\}; R \rangle$ for $\ker \hat{\phi}$, where $\langle X; R \rangle$ is a presentation for $\ker \phi$ and γ is independent of X. Since $\langle X; R \rangle$ and $\langle X'; R' \rangle$ both present the same group, Lemma 3 shows that $\langle X' \cup \{\gamma'\}; R' \rangle$ presents $\ker \hat{\phi}$. For construction (ii), by Lemma 2 there exist presentations for $\ker \phi$ and $\ker \hat{\phi}$ of the form $\langle X; R \rangle$ and $\langle X; R \cup \{r\} \rangle$, respectively, where r is not derivable from R. From Lemma 3, we see $\ker \hat{\phi}$ admits the presentation $\langle X'; R' \cup \{r'\} \rangle$, where r' is not derivable from R'.

Henceforth, we will refer to the above constructions as operations of type (i) and (ii).

Corollary. Let Γ be a finite simple graph and let $V(\Gamma)$ be partitioned into two sets A and B. Define the homomorphism $\phi: G\Gamma \to \langle t \rangle$ so that $\phi(x) = t$ for every $x \in A$ and $\phi(x) = 1$ for every $x \in B$. Then $\ker \phi$ admits a finite presentation if Γ can be constructed from a single vertex in A by applying a sequence of type (i) and type (ii) operations. Moreover, one such finite presentation of $\ker \phi$ is $\langle X_f; R_f \rangle$, where $|X_f| = |V(\Gamma)| - 1$, $|R_f| = |E(\Gamma)| - |V(\Gamma)| + 1$. In particular, if Γ is a tree and can be so constructed, then $\ker \phi$ is free on $|V(\Gamma)| - 1$ letters.

Proof. We induct on the number of steps taken to create the graph. The base case is where Γ consists of the graph on one vertex $a_0 \in A$ so that $V(\Gamma) = 1$, $E(\Gamma) = 0$, and $\ker \phi = \{1\}$.

Proceeding inductively, assume Γ is a partitioned graph constructed in the above manner and ϕ is its corresponding homomorphism, and assume there exists a presentation $\langle X_f; R_f \rangle$ for $\ker \phi$ such that $|X_f| = |V(\Gamma)| - 1$ and $|R_f| = |E(\Gamma)| - |V(\Gamma)| + 1$.

For the type (i) operation, $|V(\hat{\Gamma})| = |V(\Gamma)| + 1$ and $|E(\hat{\Gamma})| = |E(\Gamma)| + 1$. From Theorem 1, we know there exists a generating symbol γ independent of X_f such that $\langle X_f \cup \{\gamma\}; R_f \rangle$ is a presentation for $\ker \hat{\phi}$, which meets our requirements, since $|X_f \cup \{\gamma\}| = |V(\hat{\Gamma})| - 1$ and $|R_f| = |E(\hat{\Gamma})| - |V(\hat{\Gamma})| + 1$.

For the type (ii) operations, $|V(\hat{\Gamma})| = |V(\Gamma)|$ and $|E(\hat{\Gamma})| = |E(\Gamma)| + 1$. Again from Theorem 1, we know there exists a relation r' not derivable from R_f such that $\langle X; R_f \cup \{r'\} \rangle$ presents $\ker \hat{\phi}$. This presentation again meets our requirements, since $|X_f| = |V(\hat{\Gamma})| - 1$ and $|R_f \cup \{r'\}| = |E(\hat{\Gamma})| - |V(\hat{\Gamma})| + 1$. So by induction, the statement holds for all graphs that can be built by this method.

The last assertion of the corollary is immediate, since $|V(T)| = |E(T)| + 1$ for a tree T.

6. Connections. Whitehead [15] defines elementary deformations of simplicial complexes so that if K_0 and K_1 are simplicial complexes with $K_1 = K_0 + aC$, where C is a closed simplex such that $a\dot{C} \subseteq K_0$ and $C \nsubseteq K_0$, then the transformation $K_0 \to K_1$ is called an elementary expansion of order $\dim(aC)$, and the transformation $K_1 \to K_0$ is called an elementary contraction of order $\dim(aC)$.

If instead of restricting ourselves to the graph Γ, we consider the 2–skeleton $\Gamma^{(2)}$ of the simplicial complex spanned by Γ, then an operation of type (i) corresponds to an elementary simplicial expansion $K_0 \to K_0 + ax$ of order 1, where $a \in A$ and $x \in V(\Gamma)$, while an operation of type (ii) above corresponds to an elementary simplicial expansion $K_0 \to K_0 + ae$ of order 2, where $a \in A$ and $e \in E(\Gamma)$. Thus, we are building a subcomplex of $\Gamma^{(2)}$, starting with a single live vertex of A and performing elementary simplicial expansions of orders 1 and 2 in which the cone vertex a in each expansion is live.

By [15], if Γ can be constructed from a single live vertex $a_0 \in A$, by a sequence of operations of types (i) and (ii), then Γ can be constructed as $\{a_0\}\xrightarrow{(i)}T\xrightarrow{(ii)}\Gamma$, where $\xrightarrow{(i)}$ and $\xrightarrow{(ii)}$ denote a sequence of operations of type (i) and type (ii), respectively, and T is a tree. Since operations of type (ii) add no new vertices, T is a spanning tree for Γ. Notice the elementary expansions of orders 1 and 2 corresponding to $\{a_0\}\xrightarrow{(i)}T\xrightarrow{(ii)}\Gamma$ yield a contractible subcomplex of $\Gamma^{(2)}$ which contains all the edges of Γ.

The definition of a type (i) operation implies that each vertex $b \in B$ has degree 1 in T. Also, for each $b \in B$, there exists a vertex $a \in A$ which is adjacent to b in T. Finally, if we let $T_A \subseteq T$ be the subtree spanned by the vertex set A, then T_A is connected by construction since there exists a (unique) path from any vertex $a \in A$ to a_0 containing only live vertices.

For our next lemma, recall that a subgraph $\mathcal{C}$ of a graph Γ is dominating if every vertex in $V(\Gamma) - V(\mathcal{C})$ is adjacent to a vertex in $\mathcal{C}$.

Lemma 4. Let Γ be a finite simple graph with $V(\Gamma)$ partitioned into two sets A and B, and let $\mathcal{L}$ be the full subgraph spanned by A. Then $\mathcal{L}$ is a connected and dominating subgraph of Γ if and only if for all $a \in A$, there exists a spanning tree T of Γ such that $\{a\}\xrightarrow{(i)}T$.

Proof. ($\Rightarrow$) Let $a \in A$ and let T_A be a connected spanning tree for $\mathcal{L}$. Since T_A consists of live vertices, there exists a sequence of type (i) operations $\{a\}\xrightarrow{(i)}T_A$.

Since $\mathcal{L}$ is dominating, there exists a function $\lambda: B \to A$ that maps each vertex $b \in B$ to a vertex in A which is adjacent to b in Γ. Thus, λ can be used to define a sequence of type (i) operations $T_A \xrightarrow{(i)} T$, where T is a spanning tree for Γ.

($\Leftarrow$) By the previous paragraphs, the live subtree T_A is a connected and dominating subgraph of the spanning tree T of Γ. Because T_A is a spanning tree for $\mathcal{L}$, we know $\mathcal{L}$ is a connected and dominating subgraph of Γ.

Theorem 6.1 of [9] states that any nontrivial homomorphism $\chi: G\Gamma \to \langle t \rangle$ has a finitely generated kernel if and only if $\mathcal{L}(\chi)$ – the full subgraph spanned by $\{v \in V(\Gamma) : \chi(v) \neq 1\}$ — is a connected and dominating subgraph of Γ. This implies our next result.

Theorem 2. Let Γ be a finite simple graph and let $V(\Gamma)$ be partitioned into two sets A and B. Define the homomorphism $\phi: G\Gamma \to \langle t \rangle$ so $\phi(x) = t$ for every $x \in A$ and $\phi(x) = 1$ for every $x \in B$. Then $\ker \phi$ is finitely generated if and only if for all $a \in A$, there exists a spanning tree T of Γ such that $\{a\} \xrightarrow{(i)} T$.

Hence, the finite generation of $\ker \phi$ is encoded in the operations of type (i) from any live vertex of Γ to a spanning tree for Γ. However, the operations of type (i) alone do not provide enough information to determine the finite presentation of $\ker \phi$, as Example 1 shows. It follows that the finite presentation is encoded in the expansions of type (ii) in the sequence $\{a_0\} \xrightarrow{(i)} T \xrightarrow{(ii)} \Gamma$.

In [3], Droms proved a graph group $G\Gamma$ is coherent — that is, each of its finitely generated subgroups is finitely presented — if and only if each circuit of Γ of length greater than three has a chord. Hence, if Γ is chordal and for all $a \in A$, there exists a spanning tree T of Γ such that $\{a\} \xrightarrow{(i)} T$, then $\ker \phi$ is finitely presented.

Proposition. Let Γ be a finite simple graph with $V(\Gamma)$ partitioned into two sets A and B and suppose for all $a \in A$ there exists a spanning tree T of Γ such that $\{a\} \xrightarrow{(i)} T$. Then if Γ is chordal, it can be constructed from any live vertex $a \in A$ by a sequence of type (i) and type (ii) operations.

Proof. Let $a \in A$ and T be the corresponding spanning tree. Let Γ' be a graph with the maximal number of edges which contains T and that can be constructed by a sequence of type (i) and type (ii) operations from $\{a\}$. Since Γ' contains T, $V(\Gamma') = V(\Gamma)$.

Suppose $\Gamma' \neq \Gamma$. For each $e = \{x, y\} \in E(\Gamma) - E(\Gamma')$, define $l(e)$ to be the length of a minimal circuit in the graph $\Gamma' \cup \{e\}$ consisting of live vertices, with the possible exception of x and y. Such a circuit exists since Γ' contains T. Let $e_0 \in E(\Gamma) - E(\Gamma')$ be minimal with respect to this length.

Since $e_0 \notin E(\Gamma')$, we must have $l(e_0) > 3$, since otherwise e_0 could be constructed from Γ' by a type (ii) operation, contradicting the maximality of Γ'. Let

c be any minimal length circuit for e_0 as above. Then by hypothesis, c has a chord e', and e' divides c into two circuits, each of length less than c. By minimality of c, $e' \notin E(\Gamma')$. But then $l(e') < l(e_0)$, contradicting the minimality of $l(e_0)$. Thus $\Gamma' = \Gamma$ and hence, Γ can be constructed by a sequence of type (i) and type (ii) operations from $\{a\}$.

The converse of this proposition is false; the construction of Corollary 1 gives finitely presented subgroups of graph groups which are not based on chordal graphs, such as the case where Γ is the cone on a square and the cone vertex is an element of A.

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