SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

105. [1997, 105] *Proposed by Kenneth Davenport, P. O. Box 99901, Pittsburgh, Pennsylvania.*

Evaluate the series

$$\frac{3}{1}+\frac{1}{3}-\frac{1}{6}-\frac{1}{10}+\cdots$$

where the denominators are the triangular numbers and every two terms the signs alternate, i.e. $+, +, -, -, +, +$, etc.

Solution I by Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; and the proposer.

The triangular numbers, $\{T_n\}_{n=1}^{\infty}$, have the form $T_n = n(n+1)/2$. The partial sum, S_{2n}, of the first $2n$ terms of the series is

$$S_{2n} = 2 + \sum_{k=0}^{n-1}\left[\frac{2}{(2k+1)(2k+2)} + \frac{2}{(2k+2)(2k+3)}\right](-1)^k.$$

In this replace

$$\frac{1}{(2k+1)(2k+2)} \quad \text{by} \quad \frac{1}{2k+1} - \frac{1}{2k+2}$$

and

$$\frac{1}{(2k+2)(2k+3)} \quad \text{by} \quad \frac{1}{2k+2} - \frac{1}{2k+3}.$$

We obtain

$$S_{2n} = 2 + 2\sum_{k=0}^{n-1}\left[\frac{1}{2k+1} - \frac{1}{2k+3}\right](-1)^k.$$

If the summation is written out, all terms are doubled except those for $k = 0$ and $k = n - 1$. This gives

$$\begin{aligned} S_{2n} &= 2 + 2\left[2\sum_{k=0}^{n-1} \frac{1}{2k+1}(-1)^k + \frac{(-1)^n}{2n+1} - 1\right] \\ &= 4\sum_{k=0}^{n-1} \frac{1}{2k+1}(-1)^k + \frac{2(-1)^n}{2n+1}. \end{aligned}$$

The indicated summation is a partial sum of the Leibniz–Gregory series for $\tan^{-1} 1$. Hence,

$$\lim_{n\to\infty} S_{2n} = 4\sum_{k=0}^{\infty} \frac{1}{2k+1}(-1)^k = 4\left(\frac{\pi}{4}\right) = \pi.$$

Solution II by Carl Libis, University of Alabama, Tuscaloosa, Alabama.

$$\begin{aligned} \frac{3}{1} + \frac{1}{3} - \frac{1}{6} - \frac{1}{10} + \cdots &= 2\left[\frac{3}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \cdots\right] \\ &= 2\left[\left(\frac{3}{1} - \frac{3}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] \\ &= 2\left[3 - 1 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots\right] \\ &= 4\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] \\ &= 4\left[\sum_{n=0}^{\infty} \frac{1}{4n+1} - \sum_{n=0}^{\infty} \frac{1}{4n+3}\right]. \end{aligned}$$

Let $z = 1/4$ in the formula

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

We obtain

$$\begin{aligned}
\pi &= 4 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(1/4)^2 - n^2} \\
&= 4 + 8 \sum_{n=1}^{\infty} \frac{1}{1 - 16n^2} \\
&= 4 + 4 \sum_{n=1}^{\infty} \left(\frac{1}{1+4n} + \frac{1}{1-4n} \right) \\
&= 4 \left(1 + \sum_{n=1}^{\infty} \frac{1}{4n+1} - \sum_{n=1}^{\infty} \frac{1}{4n-1} \right) \\
&= 4 \left(\sum_{n=0}^{\infty} \frac{1}{4n+1} - \sum_{n=0}^{\infty} \frac{1}{4n+3} \right).
\end{aligned}$$

Therefore,

$$\frac{3}{1} + \frac{1}{3} - \frac{1}{6} - \frac{1}{10} + \cdots = \pi.$$

Solution III by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

We begin by establishing the following preliminary result.

Lemma.

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{n(n+1)/2} + \frac{1}{(n+1)(n+2)/2} + \cdots = 2.$$

Proof.

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\cdots+\frac{1}{n(n+1)/2}+\frac{1}{(n+1)(n+2)/2}+\cdots$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\cdots+\frac{1}{n(n+1)/2}+\frac{1}{(n+1)(n+2)/2}\right)$$

$$=\lim_{n\to\infty}\left[\left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{4}-\frac{2}{5}\right)+\cdots\right.$$

$$\left.+\left(\frac{2}{n}-\frac{2}{n+1}\right)+\left(\frac{2}{n+1}-\frac{2}{n+2}\right)\right]$$

$$=\lim_{n\to\infty}\left(2-\frac{2}{n+2}\right)=2.$$

Corollary.

$$1+\frac{1}{3}-\frac{1}{6}-\frac{1}{10}+\frac{1}{15}+\frac{1}{21}-\frac{1}{28}-\frac{1}{36}+\cdots$$

converges absolutely.

Recalling that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4},$$

it follows that

$$\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{4n-1}-\frac{1}{4n+1}\right)+\cdots=1-\frac{\pi}{4}.$$

Thus,

$$
\begin{aligned}
&1+\frac{1}{3}-\frac{1}{6}-\frac{1}{10}+\frac{1}{15}+\frac{1}{21}-\frac{1}{28}-\frac{1}{36}+\cdots \\
&=1+\frac{1}{3}+\left(\frac{1}{6}-\frac{1}{3}\right)+\left(\frac{1}{10}-\frac{1}{5}\right)+\frac{1}{15}+\frac{1}{21}+\left(\frac{1}{28}-\frac{1}{14}\right)+\left(\frac{1}{36}-\frac{1}{18}\right)+\cdots \\
&=\left[1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{36}+\cdots\right] \\
&\quad -\left[\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{14}+\frac{1}{18}\right)+\cdots\right] \\
&=2-4\left(\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\cdots+\frac{2}{(4n-1)(4n+1)}+\cdots\right) \\
&=2-4\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{4n-1}-\frac{1}{4n+1}\right)+\cdots\right] \\
&=2-4\left(1-\frac{\pi}{4}\right)=-2+\pi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{3}{1}+\frac{1}{3}-\frac{1}{6}-\frac{1}{10}+\cdots \\
&=2+\left(1+\frac{1}{3}-\frac{1}{6}-\frac{1}{10}+\cdots\right) \\
&=2+(-2+\pi)=\pi.
\end{aligned}
$$

106. [1997, 105] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Show that

$$\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{(n-m+1)(m+1)}=\frac{2^{n+1}-1}{(n+2)(n+1)}.$$

Solution I by Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri and Carl Libis, University of Alabama, Tuscaloosa, Alabama.

$$\binom{n}{m}=\frac{n(n-1)\cdots(n-m+1)}{m!},$$

so

$$\binom{n}{m}\frac{1}{(n-m+1)(m+1)}=\frac{1}{(n+1)(n+2)}\binom{n+2}{m+1}.$$

Hence,

$$\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{(n-m+1)(m+1)}=\frac{1/2}{(n+1)(n+2)}\sum_{m=0}^{n}\binom{n+2}{m+1}$$

$$=\frac{1/2}{(n+1)(n+2)}\sum_{m=1}^{n+1}\binom{n+2}{m}$$

$$=\frac{1/2}{(n+1)(n+2)}\left[\sum_{m=0}^{n+2}\binom{n+2}{m}-\binom{n+2}{n+2}-\binom{n+2}{0}\right]$$

$$=\frac{1/2}{(n+1)(n+2)}\left(2^{n+2}-2\right)$$

$$=\frac{2^{n+1}-1}{(n+1)(n+2)}.$$

Solution II by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

We begin by establishing the following preliminary result.

Lemma.

$$\frac{1}{n+1}\binom{n+1}{m+1} = \frac{1}{m+1}\binom{n}{m}.$$

Proof.

$$\frac{1}{n+1}\binom{n+1}{m+1} = \frac{1}{n+1}\cdot\frac{(n+1)!}{(m+1)!((n+1)-(m+1))!}$$

$$= \frac{1}{m+1}\cdot\frac{n!}{m!(n-m)!} = \frac{1}{m+1}\binom{n}{m}.$$

Corollary.

$$\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m} = \frac{1}{n+1}(2^{n+1}-1).$$

Proof.

$$\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m} = \sum_{m=0}^{n}\frac{1}{n+1}\binom{n+1}{m+1} = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m+1}$$

$$= \frac{1}{n+1}\left(\sum_{m=0}^{n+1}\binom{n+1}{m} - \binom{n+1}{0}\right) = \frac{1}{n+1}(2^{n+1}-1).$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{(n-m+1)(m+1)}\\
&=\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{n+2}\left(\frac{1}{m+1}+\frac{1}{n-m+1}\right)\\
&=\frac{1}{2}\cdot\frac{1}{n+2}\left(\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}+\sum_{m=0}^{n}\frac{1}{n-m+1}\binom{n}{n-m}\right)\\
&=\frac{1}{2}\cdot\frac{1}{n+2}\left(2\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}\right)\\
&=\frac{1}{n+2}\cdot\frac{1}{n+1}(2^{n+1}-1)=\frac{2^{n+1}-1}{(n+1)(n+2)}.
\end{aligned}
$$

Solution III by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri and Kenneth Davenport, P. O. Box 99901, Pittsburg, Pennsylvania.

First notice that

$$\sum_{m=0}^{n}\binom{n}{m}\frac{1}{(n-m+1)(m+1)}=\frac{1}{n+2}\sum_{m=0}^{n}\binom{n}{m}\left[\frac{1}{n-m+1}+\frac{1}{m+1}\right]. \tag{1}$$

Integrating the identity

$$\sum_{m=0}^{n}\binom{n}{m}x^m=(1+x)^n$$

from 0 to 1 gives

$$\sum_{m=0}^{n}\binom{n}{m}\frac{1}{m+1}=\frac{2^{n+1}-1}{n+1}. \tag{2}$$

Similarly, since

$$\sum_{m=0}^{n} \binom{n}{m} x^{n-m} = (1+x)^n,$$

$$\sum_{m=0}^{n} \binom{n}{m} \frac{1}{n-m+1} = \frac{2^{n+1}-1}{n+1}. \tag{3}$$

Substituting (2) and (3) into the right hand side of (1) and simplifying yields

$$\frac{1}{2}\sum_{m=0}^{n} \binom{n}{m} \frac{1}{(n-m+1)(m+1)} = \frac{2^{n+1}-1}{(n+2)(n+1)}.$$

Solution IV by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.

The Binomial Series is

$$\sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m} = (x+y)^n.$$

Integrating with respect to dx and dy gives

$$\sum_{m=0}^{n} \binom{n}{m} \int_0^1 x^m dx \int_0^1 y^{n-m} dy = \int_0^1 \int_0^1 (x+y)^n dx dy.$$

Therefore,

$$\sum_{m=0}^{n} \binom{n}{m} \frac{1}{(m+1)(n-m+1)} = \frac{1}{n+1}\left[\frac{(1+y)^{n+2}-y^{n+2}}{n+2}\right]_0^1 = \frac{2^{n+2}-2}{(n+2)(n+1)}.$$

Solution V by the proposer.

First we note that if

$$f(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots + a_n\frac{x^n}{n!} + \cdots$$

and

$$g(x) = b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + b_3\frac{x^3}{3!} + \cdots + b_n\frac{x^n}{n!} + \cdots ,$$

then the coefficient of $x^n/n!$ in the product $f(x)g(x)$ is

$$\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m.$$

Now, if

$$f(x) = g(x) = \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^{n-1}}{n!} + \cdots ,$$

then by the above note the coefficient of $x^n/n!$ in

$$f(x)g(x) = \frac{(e^x - 1)^2}{x^2}$$

is

$$\sum_{m=0}^{n} \binom{n}{m} \frac{1}{n-m+1} \cdot \frac{1}{m+1}. \qquad (1)$$

On the other hand,

$$f(x)g(x) = \frac{(e^x-1)^2}{x^2} = \frac{1}{x^2}(e^{2x} - 2e^x + 1)$$

$$= \left(\frac{2^2-2}{2\cdot 1}\right) + \left(\frac{2^3-2}{3\cdot 2}\right)\frac{x}{1!} + \left(\frac{2^4-2}{4\cdot 3}\right)\frac{x^2}{2!} + \cdots + \left(\frac{2^{n+2}-2}{(n+2)(n+1)}\right)\frac{x^n}{n!} + \cdots \quad (2)$$

Thus, the coefficient of $x^n/n!$ in $f(x)g(x)$ using equation (2) is

$$\frac{2^{n+2}-2}{(n+2)(n+1)}. \tag{3}$$

Consequently, from (1) and (3) we obtain the desired equality.

107. [1997, 106] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

Prove, if $p = 8k \pm 3$ is a prime for $k \geq 1$ and

$$a^2 + (p-2)b^2 \equiv 0 \pmod{p},$$

then $a \equiv 0 \pmod{p}$ and $b \equiv 0 \pmod{p}$.

Solution I by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri and Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

The congruence,

$$a^2 + (p-2)b^2 \equiv 0 \pmod{p},$$

can be rewritten as

$$a^2 + pb^2 - 2b^2 \equiv 0 \pmod{p},$$

or,

$$a^2 \equiv 2b^2 \pmod{p}.$$

Assume $p \nmid b$, and let r be a quadratic residue modulo p such that $b^2 \equiv r \pmod{p}$. Then $(r/p) = 1$, where (x/p) denotes the Legendre symbol. $a^2 \equiv 2b^2 \pmod{p}$ implies $a^2 \equiv 2r \pmod{p}$, so that $(2r/p) = 1$. But

$$(2r/p) = (2/p)(r/p) = (-1)\cdot(1) = -1,$$

since $p \equiv \pm 3 \pmod 8$ implies $(2/p) = -1$. (See [1], Section 9.2, pp. 180–187.)

Since $(2r/p)$ cannot equal both 1 and -1, this is a contradiction, so the assumption, $p \nmid b$, is false. Thus $p|b$, and $b \equiv 0 \pmod p$. $a^2 \equiv 2b^2 \pmod p$ implies $a^2 \equiv 0 \pmod p$, so that $a \equiv 0 \pmod p$, since p is prime. This completes the proof.

Reference

1. D. M. Burton, *Elementary Number Theory*, 3rd ed., McGraw–Hill, New York, 1997.

Solution II by the proposer. Let $p = 8k+3$. Because $1 = (8k+1)(4k+1) + (8k+3)(-4k)$ we have $(4k+1)(8k+1) \equiv 1 \pmod p$. Using Legendre symbols and $(4k+1)^2 = (2k+1) + 2k(8k+3)$ we have

$$\left(\frac{2k+1}{8k+3}\right) = 1$$

and

$$\left(\frac{4k+2}{8k+3}\right) = \left(\frac{2}{8k+3}\right)\left(\frac{2k+1}{8k+3}\right) = (-1)(1) = -1.$$

Suppose $a^2 + (p-2)b^2 \equiv 0 \pmod p$ and $a \not\equiv 0 \pmod p$. There exists a c such that $ac \equiv 1 \pmod p$ so $(ac)^2 + (p-2)(bc)^2 \equiv 0 \pmod p$. Therefore, $(8k+1)(bc)^2 \equiv -1 \pmod p$ and $(4k+1)(8k+1)(bc)^2 \equiv -(4k+1) \equiv 4k+2 \pmod p$. But $(bc)^2 \equiv 4k+2 \pmod p$ is a contradiction because

$$\left(\frac{4k+2}{8k+3}\right) = -1.$$

So $a \equiv 0 \pmod p$ and $b^2 \equiv 0 \pmod p$, which implies $b \equiv 0 \pmod p$.

Similarly, let $p = 8k+3$. We have $1 = (8k-5)(4k-2) + (8k-3)(-4k+3)$ implying $(4k-2)(8k-5) \equiv 1 \pmod p$. Also $(4k-1)^2 = (-2k+1) + 2k(8k-3)$ which gives

$$\left(\frac{-2k+1}{8k-3}\right) = 1$$

and

$$\left(\frac{-4k+2}{8k-3}\right) = \left(\frac{2}{8k-3}\right)\left(\frac{-2k+1}{8k-3}\right) = (-1)(1) = -1.$$

As in the preceding case there is a c such that $(p-2)(bc)^2 \equiv -1 \pmod{p}$ or $(8k-5)(bc)^2 \equiv -1 \pmod{(8k-3)}$. We have $(4k-2)(8k-5)(bc)^2 \equiv -(4k-2) \equiv -4k+2 \pmod{p}$ so $(bc)^2 \equiv -4k+2 \pmod{p}$. Again this is a contradiction so $a \equiv 0 \pmod{p}$ and $b \equiv 0 \pmod{p}$.

108. [1997, 106] *Proposed by Joseph B. Dence, University of Missouri–St. Louis, St. Louis, Missouri.*

A positive integer d is called a unitary divisor of a positive integer n, written $d\,||\,n$, if d and n/d are relatively prime. We define two unitary arithmetic functions by analogy to their standard counterparts:

A unitary Möbius function $\mu^*(n)$:

$$\sum_{d\,||\,n} \mu^*(d) = \begin{cases} 1, & \text{for } n = 1; \\ 0, & \text{for } n > 1. \end{cases}$$

A unitary Euler phi-function $\phi^*(n)$:

$$\phi^*(n) = \sum_{d\,||\,n} \mu^*(d)\frac{n}{d}.$$

When $n > 2$, $\phi(n)$ is always even; this is not true of $\phi^*(n)$. Determine how many known odd primes are in the range of the function $\phi^*(n)$.

Solution by the proposer. It is straightforward to show that $\mu^*(n)$ is multiplicative: $n = m_1m_2$, $(m_1, m_2) = 1$ implies $\mu^*(n) = \mu^*(m_1)\mu^*(m_2)$, and from this fact that $\phi^*(n)$ is also multiplicative. Next we note that if p is an odd prime and $k \geq 1$, then

$$\phi^*(p^k) = \mu^*(1)\frac{p^k}{1} + \mu^*(p^k)\frac{p^k}{p^k} = p^k - 1,$$

so if n contains one or more odd primes in its factorization, $\phi^*(n)$ is even. Hence, $\phi^*(n)$ can only be an odd prime if $n = 2^k$, in which case $\phi^*(2^k) = 2^k - 1$. Primes of this form are the Mersenne primes (e.g., 3, 7, 31, 127, ...); at present, there are approximately three dozen known Mersenne primes.