SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

104. [1997, 35; 1998, 59–61] *Proposed by Kenneth Davenport, Box 491, Frackville, Pennsylvania.*

Show that

$$1 \cdot \sin\frac{\pi}{2n} + 3 \cdot \sin\frac{3\pi}{2n} + 5 \cdot \sin\frac{5\pi}{2n} + \cdots + (2n-1)\sin\frac{(2n-1)\pi}{2n} = n\csc\frac{\pi}{2n}.$$

Solution III by Paul S. Bruckman, Edmonds, Washington.

Let

$$o_k = \frac{1-(-1)^k}{2}.$$

Then,

$$\sum_{k=0}^{2n-1} k\sin\frac{k\pi}{2n} \cdot o_k = n \cdot \csc\frac{\pi}{2n}, \quad n = 1, 2, \ldots$$

is easily evaluated using complex variables. That is,

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right),$$

and then we can use geometric series sum formulas to evaluate the sum. Letting $\theta = \exp(i\pi/2n)$ and using the fact that $\theta^{2n} = -1$, we have

$$\sum_{k=0}^{2n-1} k \sin \frac{k\pi}{2n} \cdot o_k = \frac{1}{2i} \sum_{k=0}^{2n-1} k o_k \left(\theta^k - \theta^{-k}\right)$$

$$= \frac{1}{2i} \sum_{k=0}^{2n-1} k o_k \theta^k - \frac{1}{2i} \sum_{k=1}^{2n} (2n-k) o_{2n-k} \theta^{k-2n}$$

$$= \frac{1}{2i} \sum_{k=0}^{2n-1} k o_k \theta^k + \frac{1}{2i} \sum_{k=0}^{2n-1} (2n-k) o_k \theta^k = \frac{n}{i} \sum_{i=0}^{2n-1} o_k \theta^k$$

$$= \frac{n}{2i} \left(\frac{\theta^{2n}-1}{\theta - 1} - \left(\frac{(-\theta)^{2n} - 1}{-\theta - 1} \right) \right) = \frac{n}{2i} \left(\frac{-2}{\theta - 1} + \frac{-2}{\theta + 1} \right)$$

$$= \frac{-n}{i} \cdot \frac{2\theta}{\theta^2 - 1} = n \cdot \frac{2i}{\theta - \theta^{-1}} = n \csc \frac{\pi}{2n}.$$

109. [1997, 184] *Proposed by Kenneth Davenport, Box 491, Frackville, Pennsylvania.*

Let n be a positive integer and $a \geq 2$ be a positive integer. Show that

$$\int_0^\infty \frac{dx}{1^a + x^a} + \int_0^\infty \frac{dx}{2^a + x^a} + \cdots + \int_0^\infty \frac{dx}{n^a + x^a}$$

$$= \left[\frac{1}{1^{a-1}} + \frac{1}{2^{a-1}} + \cdots + \frac{1}{n^{a-1}} \right] \frac{\pi/a}{\sin(\pi/a)}.$$

Solution I by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

Consider

$$\int_C f(z) dz$$

where $f(z) = 1/(m^a + z^a)$, $m = 1, 2, \ldots, n$ and C is the boundary of the circular sector described by $0 \leq \theta \leq 2\pi/a$ and $0 \leq r \leq R$.

The only singularity of $f(z)$ that is inside C is a simple pole at $z_0 = me^{\pi i/a}$ and

$$\mathrm{Res}_{z=z_0} f(z) = -e^{\pi i/a}/(am^{a-1}).$$

By the Residue Theorem,

$$\int_0^R \frac{dx}{m^a + x^a} + \int_0^{2\pi/a} \frac{Re^{i\theta} i d\theta}{m^a + R^a e^{ia\theta}} + \int_R^0 \frac{e^{2\pi i/a} dr}{m^a + r^a} = \frac{-2\pi i e^{\pi i/a}}{am^{a-1}}. \tag{1}$$

Let $I = I(R)$ represent the second integral on the left hand side of (1). It is straightforward to show that

$$|I| \leq \frac{2\pi R}{a|R^a - m^a|}.$$

Since $a \geq 2$, $\lim_{R\to\infty} I(R) = 0$. Then (1) becomes

$$(1 - e^{2\pi i/a}) \int_0^\infty \frac{dx}{m^a + x^a} = \frac{-2\pi i e^{\pi i/a}}{am^{a-1}}. \tag{2}$$

Equating real parts in (2) gives

$$\left(1 - \cos\frac{2\pi}{a}\right) \int_0^\infty \frac{dx}{m^a + x^a} = \frac{2\pi \sin\frac{\pi}{a}}{am^{a-1}}$$

and so

$$\int_0^\infty \frac{dx}{m^a + x^a} = \frac{2\pi \sin(\pi/a)}{2am^{a-1}\sin^2(\pi/a)} = \frac{\pi}{am^{a-1}\sin(\pi/a)}.$$

Therefore,

$$\sum_{m=1}^n \int_0^\infty \frac{dx}{m^a + x^a} = \frac{\pi/a}{\sin(\pi/a)} \sum_{m=1}^n \frac{1}{m^{a-1}}.$$

Solution II by Joseph E. Chance, University of Texas–Pan American, Edinburg, Texas.

For $i = 1, 2, \ldots, n$,

$$\int_0^\infty \frac{dx}{i^a + x^a} = \frac{1}{i^a} \int_0^\infty \frac{dx}{1 + (x/i)^a}.$$

Let $y = x/i$ and the integral is transformed to

$$\frac{1}{i^{a-1}} \int_0^\infty \frac{dy}{1 + y^a}.$$

The required sum can now be written as

$$\left[\frac{1}{1^{a-1}} + \frac{1}{2^{a-1}} + \cdots + \frac{1}{n^{a-1}}\right] \int_0^\infty \frac{dy}{1 + y^a}.$$

In this integral let

$$u = \frac{1}{1 + y^a}, \quad \text{or} \quad y = \left(\frac{1-u}{u}\right)^{\frac{1}{a}}, \quad dy = \frac{1}{a}\left(\frac{1-u}{u}\right)^{\frac{1}{a}-1} \left(\frac{-1}{u^2}\right)$$

which transforms the integral to

$$\frac{1}{a} \int_0^1 (1-u)^{\frac{1}{a}-1} u^{-\frac{1}{a}}\, du.$$

In terms of the famous beta function, the integral is

$$\frac{1}{a} B\left(1 - \frac{1}{a}, \frac{1}{a}\right),$$

but in terms of the more famous gamma function and its identities, this beta function is

$$\frac{1}{a}\frac{\Gamma\left(\frac{a-1}{a}\right)\Gamma\left(\frac{1}{a}\right)}{\Gamma\left(\frac{a-1}{a}+\frac{1}{a}\right)} = \frac{1}{a}\Gamma\left(1-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}\right) = \frac{1}{a}\frac{\pi}{\sin(\pi/a)}.$$

Solution III by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Carl Libis, University of Alabama, Tuscaloosa, Alabama, Jerry Masuda, Metropolitan Community College, Blue Springs, Missouri; and Ice B. Risteski, Skopje, Macedonia.

According to Formula 402 on p. 303 of the Eleventh Edition of the *C. R. C. Standard Mathematical Tables* for $0 < m < a$,

$$\int_0^\infty \frac{t^{m-1}dt}{1+t^a} = \frac{\pi}{a\sin\frac{m\pi}{a}}. \tag{$*$}$$

Let k be an arbitrary fixed positive integer and let $a \geq 2$ be a positive integer. Then, letting $m = 1$ and $t = x/k$ in $(*)$, we have

$$\int_0^\infty \frac{dx}{k^a+x^a} = \frac{1}{k^a}\int_0^\infty \frac{dx}{1+\left(\frac{x}{k}\right)^a} = \frac{k}{k^a}\int_0^\infty \frac{\frac{1}{k}dx}{1+\left(\frac{x}{k}\right)^a}$$

$$= \frac{1}{k^{a-1}}\int_0^\infty \frac{d\left(\frac{x}{k}\right)}{1+\left(\frac{x}{k}\right)^a} = \frac{1}{k^{a-1}}\frac{\pi}{a\sin\frac{\pi}{a}}$$

$$= \frac{1}{k^{a-1}}\frac{\frac{\pi}{a}}{\sin\left(\frac{\pi}{a}\right)}.$$

Thus,

$$\int_0^\infty \frac{dx}{1^a+x^a} + \int_0^\infty \frac{dx}{2^a+x^a} + \cdots + \int_0^\infty \frac{dx}{n^a+x^a}$$

$$= \frac{1}{1^{a-1}}\frac{\left(\frac{\pi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)} + \frac{1}{2^{a-1}}\frac{\left(\frac{\pi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)} + \cdots + \frac{1}{n^{a-1}}\frac{\left(\frac{\pi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)}$$

$$= \left[\frac{1}{1^{a-1}} + \frac{1}{2^{a-1}} + \cdots + \frac{1}{n^{a-1}}\right]\frac{\left(\frac{\pi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)}.$$

Also partially solved by the proposer.

110. [1997, 184] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let α, β, and γ be the three angles of any triangle. Show that

$$\frac{\sin\alpha}{1+\sin\beta\sin\gamma}+\frac{\sin\beta}{1+\sin\alpha\sin\gamma}+\frac{\sin\gamma}{1+\sin\alpha\sin\beta}<2.$$

Solution I by the proposer.

Without loss of generality, we may assume that $\sin\alpha \leq \sin\beta \leq \sin\gamma$. Now, from the fact that $0 \leq (1-\sin\alpha)(1-\sin\beta)$, we have

$$\sin\alpha+\sin\beta \leq 1+\sin\alpha\sin\beta < 1+2\sin\alpha\sin\beta.$$

On the other hand,

$$\sin\alpha+\sin\beta+\sin\gamma \leq 1+\sin\alpha+\sin\beta < 2+2\sin\alpha\sin\beta = 2(1+\sin\alpha\sin\beta).$$

Therefore,

$$\begin{aligned}
&\frac{\sin\alpha}{1+\sin\beta\sin\gamma}+\frac{\sin\beta}{1+\sin\alpha\sin\gamma}+\frac{\sin\gamma}{1+\sin\alpha\sin\beta}\\
&\leq \frac{\sin\alpha}{1+\sin\alpha\sin\beta}+\frac{\sin\beta}{1+\sin\alpha\sin\beta}+\frac{\sin\gamma}{1+\sin\alpha\sin\beta}\\
&= \frac{\sin\alpha+\sin\beta+\sin\gamma}{1+\sin\alpha\sin\beta} < \frac{2(1+\sin\alpha\sin\beta)}{1+\sin\alpha\sin\beta}\\
&= 2.
\end{aligned}$$

Solution II by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.

A generalization is given as Problem 25, p. 34 of *Leningrad Math Olympiads* (1987–1991) by D. Fomin and A. Kirichenko, MathPro Press, 1994.

Problem. If $0 \le A, B, C \le 1$, prove

$$\frac{A}{1+BC} + \frac{B}{1+AC} + \frac{C}{1+AB} \le 2.$$

Solution. Without loss of generality, assume $0 \le A \le B \le C \le 1$. Since $0 \le (1-A)(1-B)$, we have $A+B \le 1+AB \le 1+2AB$. Furthermore, $A+B+C \le A+B+1 \le 2+2AB = 2(1+AB)$. Hence,

$$\frac{A}{1+BC} + \frac{B}{1+AC} + \frac{C}{1+AB} \le \frac{A}{1+AB} + \frac{B}{1+AB} + \frac{C}{1+AB} = \frac{A+B+C}{1+AB} \le 2.$$

Note that in the original problem, only one sin can be 1 (angle $= \pi/2$) so we have strict inequality.

111. [1997, 185] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

D is a 3 by 3 determinant whose elements are polygonal numbers $P_{n,k}$ such that

$$a_{i,j} = P_{n+3i+j-4,k}, \quad k \ge 3,$$

where $P_{n,k}$ is the nth polygonal number of k "dimensions" ($P_{5,3}$ is the 5th triangular number). Show that D is a cube independent of n.

Solution by Ice B. Risteski, Skopje, Macedonia.

The formula for the n–th polygonal number of k "dimensions" is given by $P_{n,k} = nr/2$, where $r = pn + q$, $p = k-2$ and $q = -k+4$, (see [1]). Now, the 3×3 determinant D whose entries are $a_{i,j} = P_{n+3i+j-4,k}$, $(k \ge 3)$ has a form

$$D = \frac{1}{8}\det\begin{pmatrix} nr & (n+1)(r+p) & (n+2)(r+2p) \\ (n+3)(r+3p) & (n+4)(r+4p) & (n+5)(r+5p) \\ (n+6)(r+6p) & (n+7)(r+7p) & (n+8)(r+8p) \end{pmatrix}. \qquad (1)$$

Let R_1, R_2, and R_3 denote the rows of determinant (1). Then, we easily obtain the linear combination

$$R_1 - 2R_2 + R_3 = 18p\,(1 \quad 1 \quad 1)\,.$$

Hence, (1) will become

$$D = \frac{9p}{4} \det \begin{pmatrix} 1 & 1 & 1 \\ (n+3)(r+3p) & (n+4)(r+4p) & (n+5)(r+5p) \\ (n+6)(r+6p) & (n+7)(r+7p) & (n+8)(r+8p) \end{pmatrix}$$

$$= \frac{9p}{4} \det \begin{pmatrix} np+r+7p & np+r+9p \\ np+r+13p & np+r+15p \end{pmatrix} = (6-3k)^3.$$

Reference

1. J. T. Bruening, Solution to Problem 97, *Missouri Journal of Mathematical Sciences*, 9 (1997), 189–191.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; Carl Libis, University of Alabama, Tuscaloosa, Alabama; Kenneth B. Davenport, Box 491, Frackville, Pennsylvania; and the proposer.

112. [1997, 185] *Proposed by Mathew Timm, Bradley University, Peoria, Illinois.*

Let Y be a connected, first countable Hausdorff space. Then Y is h–connected if and only if where $p: X \to Y$ is a finite–sheeted covering projection from a connected space X onto Y, it follows that X is homeomorphic to Y. Y is trivially h–connected if and only if whenever $p: X \to Y$ is a connected finite–sheeted covering projection of X onto Y, it follows that p is a homeomorphism of X onto Y.

Note that for non–trivially h–connected spaces, the covering projection $p: X \to Y$ is not required to be a homeomorphism, only that some homeomorphism exist between X and Y. Examples of non–trivially h–connected spaces include the circle S^1, the torus $S^1 \times S^1$, and, more generally, the n–tori $S^1 \times \cdots \times S^1$. Examples of trivially h–connected spaces include any simply connected finite simplicial complex or, more generally, any finite simplicial complex whose fundamental group has no proper finite index subgroups.

Recall that a topological space Y has the fixed point property if and only if, for every continuous function $f: Y \to Y$, there is a $y \in Y$ such that $f(y) = y$.

Now assume that Y is a first countable, Hausdorff, connected, locally path connected, semi–locally 1–connected space. Show that if Y is h–connected and has the fixed point property, then Y is trivially h–connected.

Solution by the proposer.

Since Y is connected, locally path connected, and semi–locally 1–connected, it follows from [2, 2.5.13] that for each subgroup $H \le \pi_1(Y)$, there is a covering projection $p: X \to Y$ of a connected, locally path connected, semi–locally 1–connected X onto Y such that $p_*(\pi_1(X)) = H$.

Now assume that Y is h–connected and has the fixed point property. In addition, assume that Y is non–trivially h–connected. Then there is an n–to–1 covering projection $p: X \to Y$ for some $1 < n < +\infty$. The space X satisfies the conditions listed in the first paragraph of the solution. In particular, X is connected and $[\pi_1(Y) : p_*(\pi_1(X))] = n > 1$ is finite. Therefore, by standard results in group theory, e.g., [1, 3.3.5], there is a normal subgroup $N \le \pi_1(Y)$, called the normal core of $p_*(\pi_1(X))$, such that $N \le p_*(\pi_1(X))$ and $n \le [\pi_1(Y) : N] = k < +\infty$.

By the remarks in the first paragraph of this solution, there is a covering projection $q: \tilde{Y} \to Y$ such that $\tilde{Y}$ is connected and $q_*(\tilde{Y}) = N$. Since N is normal in $\pi_1(Y)$, it follows from [2, 2.6.2] that $q: \tilde{Y} \to Y$ is a regular covering and has a group of covering transformations $\mathrm{Aut}_Y \tilde{Y} \cong \pi_1(Y)/N$.

Now let $f \in \mathrm{Aut}_Y \tilde{Y}$. Since $\tilde{Y}$ is a connected k–to–1 covering of Y and Y is h–connected it follows that $\tilde{Y}$ is homeomorphic to Y. Since Y has the fixed point property, so does $\tilde{Y}$. Therefore, f has a fixed point, y_0. So, by [2, 2.6.5] and the paragraph following [2, 2.6.6], it follows that $f = \mathrm{id}_{\tilde{Y}}$. Thus $\mathrm{Aut}_Y \tilde{Y} \cong \pi_1(Y)/N = 1$. So $N = \pi_1(Y)$. So, since $N \le p_*(\pi_1(X))$, it also follows that $p_*(\pi_1(X)) = \pi_1(Y)$. Thus, p is a 1–1 covering projection and so it follows that p is a homeomorphism. Thus, Y is trivially h–connected.

An inspection of the above proof points out that the full power of the fixed point property is not needed. All that is required is that homeomorphisms from Y to itself have the fixed point property.

References

1. W. R. Scott, *Group Theory*, Dover, 1987.
2. E. H. Spanier, *Algebraic Topology*, McGraw–Hill, 1996.