ON THE LENGTH OF A CIRCULAR ARC

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In his note on limits [1], Professor Krantz points out that the limit

$$(*) \qquad \lim_{x \to 0} \frac{\sin x}{x}$$

is calculated in the same way in every calculus book that he knows of (and that we know of). The upper estimate

$$\frac{\sin x}{x} \leq 1$$

is found by an obvious comparison of a chord against an arc of a circle and the lower estimate

$$\frac{\sin x}{x} \geq \cos x$$

is computed using the area of a circle, which depends on knowing the limit $(*)$ itself. To avoid this circular argument, he presented another lower estimate,

$$\frac{\sin x}{x} \geq \frac{1}{1 + \tan x}.$$

The purpose of this note is to produce one simple proof for both the inequalities

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for} \quad 0 < x < \frac{\pi}{2},$$

using only the definition of length of arc and the obvious comparisons of linear segments of appropriate partitions of the lengths involved.

Figure 1.

For every partition $\{P_i\}$ of the circular arc AB of length $x < \frac{\pi}{2}$ on the unit circle, consider the corresponding partitions $\{Q_i\}$ and $\{R_i\}$ of the linear segments AC and BD, respectively, as in Figure 1. Obviously,

$$|R_i R_{i+1}| \leq |P_i P_{i+1}| \leq |Q_i Q_{i+1}|,$$

where $|XY|$ denotes the length of the linear segment XY. Summing up and taking least upper bounds of the collections of sums, we get

$$\sin x \leq x \leq \tan x$$

and finally

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

<u>*Reference*</u>

1. S. G. Krantz, "On the Area Inside a Circle," *Missouri Journal of Mathematical Sciences*, 4 (1992), 2–8.

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