FROM TWIN–PEAKS TO MULTIPEAKS: AN APPLICATION OF JENSEN'S INEQUALITY

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1. Introduction. We shall begin with the following question.

<u>Problem 1</u>. One is given a circular disk of radius 1 and asked to cut out a sector of angle θ that will be used to form a cone. How large should the angle θ be in order to construct a cone of maximum volume?

This is a standard calculus problem which is accessible to first semester calculus students. Suppose that $x = \theta/2\pi$, then $0 \le x \le 1$. From the following Figure 1, we see that the volume of the cone made by a sector of angle θ can be written as

$$V(x) = \frac{\pi}{3}x^2(1-x^2)^{1/2}.$$

Figure 1.

By differentiating V with respect to x we find that $V(x)$ has two critical points 0 and $\sqrt{2/3}$. The second derivative test indicates that $V(x)$ attains its maximum at $\sqrt{2/3}$, that is, $V(\sqrt{2/3}) = 2\sqrt{3}\pi/27$.

Let us note that, in Problem 1, the remaining part of the disk can be used to make another cone. This leads us to the following question.

<u>Problem 2</u>. A circular disk of radius 1 is cut into two sectors, each of which is used to form a cone. How do we make the cut in order to minimize the total volume of the two cones?

Figure 2.

Although this is again a simple first semester calculus problem, the output reflects some interesting symmetric properties of the underlying function which enlightens us to the main problem of this note. We shall let x be the same as in

Problem 1 and denote the volume of the first cone by $V_1(x)$. Then the volume of the second cone is $V_2 = V_1(1-x)$. The total volume of the two cones is given by

$$V(x) = V_1(x) + V_1(1-x) = \frac{\pi}{3}\{x^2(1-x^2)^{1/2} + (1-x)^2[1-(1-x)^2]^{1/2}\},$$

the quantity we want to minimize. Instead of differentiating the right hand side of the above equation, we notice that $V(x)$ is symmetric with respect to $x = 1/2$. Therefore,

$$V'(x) = V_1'(x) - V_1'(1-x), \text{ and } V'(1/2) = 0.$$

Figure 3.

Following the second derivative test we find that $V(x)$ attains a local minimum at $1/2$. The two problems above together with some relatives have been discussed by Adrian Oldknow in [8]. It is interesting to see that the "twin–peaks" can refer to the two cones, and also can be regarded as the two peaks on the graph of the volume function $V(x)$. See Figure 3. This vivid example brought my attention to the general problem in the following section.

2. Multivariable Functions and Multipeaks. We now are concerned with the following question.

Problem 3. One is asked to cut a circular disk of radius 1 into n ($n \geq 3$) sectors of angles $\theta_1, \theta_2, \ldots, \theta_n$ and use these to form n cones, where $0 < \theta_i < \pi$, $i = 1, 2, \ldots, n$; and $\sum_{i=1}^{n} \theta_i = 2\pi$. How do we make the cuts in order to have the minimal total volume for the n cones?

Let $V_i(x_i)$ be the volume of the cone constructed from the sector of angle θ_i, where $x_i = \theta_i/2\pi$, $i = 1, 2, \ldots, n$. Then the total volume of the n cones is

$$V(x) = V(x_1, x_2, \ldots, x_n) = \frac{\pi}{3} \sum_{i=1}^{n} x_i^2 (1 - x_i^2)^{1/2},$$

where $x_i \in (0, 1/2)$ and $\sum_{i=1}^{n} x_i = 1$.

Suggested by Problem 2, we claim that $V(x)$ attains its minimum at the point $(1/n, 1/n, \ldots, 1/n)$. In general, one could find the extreme value of $V(x)$ under the constraint $\sum_{i=1}^{n} x_i = 1$ by using the method of Lagrange multipliers. However, when the objective function involves more than three variables, the routine calculation could cause certain degrees of complexity. First of all, locating the critical point(s) requires solving a system of equations. Then one needs to compute a number of determinants of different orders at each critical point to decide where the function attains local minimum and local maximum. Finally, perhaps, a more difficult task is to find the global extremum [5]. The interested readers are invited to carry out the necessary steps described above and compare the amount of calculation with the solution in this note. In what follows, we shall apply the well–known Jensen's inequality for a convex function to this problem. By a brief review of convex functions, we can solve the general problem easily and perceive the extreme value problem from a different viewpoint.

Convex Functions. Let $I = (a, b)$, $\overline{I} = [a, b]$, and f be a real function defined on I or $\overline{I}$. A function f is called *convex* (resp. *strictly convex*) on a segment $\overline{I}$ if and only if

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2) \qquad \text{(resp., <)}$$

holds for all $x_1, x_2 \in \overline{I}$, and for all real numbers $\lambda \in [0, 1]$. In particular, when $\lambda = 1/2$, and f is strictly convex, we have

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} \tag{1}$$

with equality holding if and only if $x_1 = x_2$.

This inequality is known as Jensen's inequality for strictly convex functions [6,7]. When f is twice differentiable on I, then f is convex (resp., strictly convex) if and only if $f'' \geq 0$ (resp., $>$). Intuitively, the graph of a strictly convex function f is always below the chord segment joining every pair of its points. See Figure 4.

Figure 4.

In general, Jensen's inequality, one of the most important and best known classical analytic inequalities states that if f is convex on $\overline{I}$, then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i), \tag{2}$$

for any $x_i \in \overline{I}$, $i = 1, 2, \ldots, n$ and for any $\lambda_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$ [1,6]. When $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/n$, and f is strictly convex, then inequality (2) becomes

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{3}$$

and equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

We shall omit the proof of (2) and (3), since it can be found in many references such as [4,6,7]. Under most circumstances, students would be easily convinced by the special case when $n = 2$ and the above Figure 4. The rigorous proof of the general case is a matter of induction.

Solution of Problem 3. We shall apply inequality (3) to the total volume function. Let $f(x) = \frac{\pi}{3}x^2(1 - x^2)^{1/2}$, then $V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f(x_i)$. We want to show that

$$\frac{\pi}{3}\frac{\sqrt{n^2 - 1}}{n^2} = nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} f(x_i) = V(x_1, x_2, \ldots, x_n).$$

To this end, by virtue of Jensen's inequality (3), we only need to verify that f is a strictly convex function on $(0, 1/2)$, or $f''(x) > 0$ for $x \in (0, 1/2)$. A direct calculation yields that

$$f'(x) = \frac{\pi}{3}\frac{x}{\sqrt{1-x^2}}(2-3x^2), \quad \text{and}$$

$$f''(x) = \frac{\pi}{3}\frac{1}{\sqrt{1-x^2}}\left[\frac{2-3x^2}{1-x^2} - 6x^2\right].$$

It is clear that $f''(x) > 0$ for $x \in (0, 1/2)$ if and only if

$$6x^4 - 9x^2 + 2 > 0 \quad \text{on} \quad (0, 1/2).$$

By either solving the above quadratic inequality in x^2 directly or graphing the function $6x^4 - 9x^2 + 2$ on a graphing calculator, it is not hard to find that $f''(x) > 0$ on $(0, 1/2)$. Therefore, the total volume function $V(x_1, x_2, \ldots, x_n)$ attains its global minimum at the point $(1/n, \ldots, 1/n)$, that is,

$$V(1/n, 1/n, \ldots, 1/n) = \frac{\pi}{3}\sqrt{n^2-1}/n^2 \leq V(x_1, x_2, \ldots, x_n).$$

3. Concluding Remarks.

(1) Problem 3 requires that $x_i \in (0, 1/2)$ for $i = 1, 2, \ldots, n$, that is, each sector cannot be bigger than one half of the disk. For $n = 2$, this is impossible except that $x_1 = x_2 = 1/2$. One of the differences between the cases of $n = 2$ and $n \geq 3$ is that, $V(1/2)$ in [8] is only a local minimum (intuitively, if x_1 is small enough, the resulting sum of the two cones could be very small) whereas our $V(1/n, \ldots, 1/n)$ with $n \geq 3$ is the global minimum (see Figure 5 below). Nevertheless, the formula $V(1/n, 1/n, \ldots, 1/n) = \frac{\pi}{3}\frac{\sqrt{n^2-1}}{n^2}$ is a natural generalization of $V(1/2) = \frac{\pi}{3}\frac{\sqrt{3}}{4} \approx 0.45345$ in [8]. By contrast with Oldknow's twin peaks in his flat function, we include a graph of $V(x)$ when $n = 3$. Keep in mind that the variables x_1, x_2, x_3 are in $(0, 1/2)$ and constrained by $\sum_{i=1}^{3} x_i = 1$. Therefore, only the part of the graph over the region $\frac{1}{2} - x_1 < x_2$ represents our problem. See Figure 5.

Figure 5.

Note that the three maxima at the corners might be viewed as "tri–peaks." An alternating interpretation of the term "tri–peaks" would be the three resulted cones.

(2) My motivation to this problem is very much inspired by Chakerian's note [2] which not only generalized the earlier work of Kouba [3], but also brought up some very interesting geometric concepts, as well as some different approaches to extremum problems in standard calculus. My primary intention in this note is to follow Chakerian's step in the same direction. For more extremum problems using different inequalities, we recommend the interested reader to Ivan Niven's book *Maxima and Minima Without Calculus* published by the MAA in 1981.

(3) To conclude this note, we shall present the following theorem whose proof is the same as for Problem 3.

Theorem. Let $x_i \in (0, l)$, $i = 1, 2, \ldots, n$; and $\sum_{i=1}^{n} x_i = k$, where $k \in (0, 2l)$, $l = \{(1 - 1/2p) - [(1 - 1/2p)^2 - (p-1)/(p+1)]^{1/2}\}^{1/p}$, and $p > 1$ are constants. Then

$$\sum_{i=1}^{n} x_i^p (1 - x_i^p)^{1/p} \geq \frac{k^p [n^p - k^p]^{1/p}}{n^p},$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$.

Proof. The function $f(x) = x^p (1 - x^p)^{1/p}$ $(p > 1)$ is strictly convex (i.e., $f'' > 0$) on the interval $(0, l)$, and the desired result follows from Jensen's inequality (3).

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