THE TOWER OF HANOI PROBLEM AND MATHEMATICAL THINKING

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Too often students leave a course in mathematics with the impression that mathematics can only be done in a statement–proof style, and that only especially gifted people are capable of originating such work. They fail to appreciate that conjecturing is an integral part of the mathematical process, and that this does not necessarily require extraordinary talent. How can this false impression be altered? One way is by allowing students to see that they themselves are able to make conjectures on a substantial mathematical problem. Initial conjectures can be modified or discarded if further checking shows them to be incorrect. Verification of the conjecture involves its proof. By carrying out, either individually or in a group, both parts of the mathematical process–conjecture and proof, students gain a far better understanding of mathematical thinking. However, conjecturing is in itself a valuable exercise, even if it is not combined with proving the conjecture.

In this paper the solution of a problem related to the Tower of Hanoi problem is given. The proof only uses mathematical induction, so it is within the reach of many students. In the Tower of Hanoi problem there are three poles and some rings, no two the same size. Initially, the rings are placed on a single pole A. Neither initially nor during play is a larger ring ever above a smaller ring. A move consists of taking the top ring from one pole and placing it on another pole. The Tower of Hanoi problem is then to find the least number of moves required to move all rings from the "initial pole" A to the "terminal pole" C. The other pole B will be called the "intermediate pole."

A *position* will mean a permitted arrangement of the rings on the poles *with the poles labeled.* The problem is the following: **how can one tell whether a particular position actually occurs during the transfer of rings that takes place in the Tower of Hanoi problem?** At this point the reader might try to find the correct conjecture himself. The only approach appears to be the empirical one, that is, examining the actual positions which occur. It is interesting to ask how many cases students examine on the average before the correct conjecture is found.

Two numbers have the same parity if they are either both even or both odd, otherwise they have opposite parity. To state the conjecture concisely, let's agree that there are n rings and that they are numbered in increasing size from 1 to n, n being the largest ring. Here then is the correct conjecture.

A position is attained during the transfer process if and only if the following two conditions are satisfied:

i) any two adjacent rings (on the same pole) have opposite parity; and
ii) a ring at the bottom of a pole has the same parity as n if it is either on A or C, and it has opposite parity to n if it is on the intermediate pole B.

It should be expected that a preliminary conjecture will consist of only one of the conditions above. Indeed, the author first made the preliminary conjecture that a position occurs if and only if (i) holds.

In the Tower of Hanoi problem the largest ring n is moved only once. The transfer process involves three stages: first, the smallest $n-1$ rings are transferred to the intermediate pole B, then ring n is moved from pole A to pole C, and, lastly, the smallest $n-1$ rings are transferred from pole B to pole C. The total number of moves is $2^n - 1$, but this information is not actually needed. In the inductive proofs below only the inductive step will be discussed. We leave to the reader the verification of the case $n = 1$.

The necessity of ii) is easily proved by induction. It follows from the inductive hypothesis that during the first stage of the transfer, the bottom ring on B (which can be considered the terminal pole for the transfer of the $n-1$ smallest rings) has the same parity as $n-1$ while the bottom ring on C has the opposite parity to $n-1$. During the third stage of the process the bottom ring on B (which can now be considered the initial pole for the transfer of the $n-1$ rings) must still have the same parity as $n-1$, while the bottom ring on A must have parity opposite to $n-1$.

In proving the necessity of i), the inductive hypothesis implies that any two adjacent rings, neither of which is n, must have opposite parity. Thus, it suffices to check that any ring which lands on the largest ring n has parity opposite to n. However, by ii), while ring n remains on pole A, the ring above it has the same parity as $n-1$ (since pole A is the initial pole for the transfer of the smallest $n-1$ rings during the first stage). A similar argument holds for the case in which ring n has already been transferred to C.

Next, it must be proved that conditions i) and ii) are sufficient, that is, any position satisfying these conditions actually occurs. Consider a particular position of n rings satisfying i) and ii). Ring n must either be on the initial pole A or the terminal pole C. Consider for now the case that it is on the initial pole. Recall that this occurs during the first stage of the transfer process, that is, when the smallest $n-1$ rings are being transferred from pole A to pole B. The idea of the proof is this: consider the position of the $n-1$ smallest rings obtained from the particular position of the n rings by simply ignoring ring n, and show that *this* position satisfies both conditions i) and ii). It will then follow from the inductive hypothesis that this position of the $n-1$ smallest rings actually occurs, keeping in mind that for the transfer of the $n-1$ rings pole A is the initial pole and pole B is the terminal pole.

The position of the smallest $n-1$ rings clearly satisfies condition i) since the position of the n rings satisfies i). Since the n rings satisfy i) and ring n is on pole A the second largest ring on A must have parity opposite to n. Since ii) is satisfied, the largest ring on C has the same parity as n, and the largest ring on B has parity opposite to n. But this says that if ring n is ignored, the largest rings on A and B have the same parity as $n-1$, and the largest ring on C has parity opposite to $n-1$. That is, the position of the $n-1$ smallest rings satisfies both i) and ii), if A and B are the initial and terminal poles, respectively. But these poles are just for the transfer during the first stage, so this position must occur by the inductive hypothesis! The argument for the case in which ring n is on pole C is similar.

There are other interesting questions connected to the Tower of Hanoi problem. For instance, given a position which is attained during play, what is an efficient method to decide what the next move should be? Another interesting problem, posed and solved by my colleague Kent Merryfield, is the determination of the number of distinct arrangements on a particular pole occurring during play in which there are n rings altogether. The empty pole is included in this count. This number will be the same for the terminal and initial poles, and the number of arrangements on the intermediate pole for n rings equals the number of arrangements on the initial (or terminal) pole for $n-1$. In fact, the value σ_n of the number of arrangements of n rings on the initial pole satisfies the Fibonacci recursion formula

$$\sigma_{n+2} = \sigma_{n+1} + \sigma_n$$

with initial values $\sigma_1 = 2$ and $\sigma_2 = 3$. This can be proved by using the characterization of the positions given above.

The author wishes to acknowledge his friend David Kantor for bringing this problem to his attention (over twenty years ago!).

Editorial Note: The interested reader might like to refer to "Pascal's Triangle and the Tower of Hanoi," by Andreas M. Hinz (*American Mathematical Monthly*, June–July 1992, pp. 538–544) for another treatment of the above topic.

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