ON WEAKLY LPN RINGS

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Introduction. All rings in this paper are assumed to be associative with identity. By an Artinian (Noetherian) ring, we mean a ring that is both left and right Artinian (Noetherian). For any ring R, $P(R)$ and $J(R)$ will denote its prime and Jacobson radicals, respectively. R is said to be a left perfect ring if $R/J(R)$ is Artinian and $J(R)$ is left T–nilpotent. If $R/J(R)$ is Artinian and idempotents of $R/J(R)$ can be lifted to R, we say that R is semiperfect. It is well–known that all left perfect rings are semiperfect but not vice versa. In addition, it can be shown (see Proposition 1.4) that if R is left perfect then the following conditions are equivalent:

(i) $R/P(R)$ is Artinian and $P(R)$ is left T–nilpotent;
(ii) $R/P(R)$ is Artinian and $J(R)$ is left T–nilpotent.

In this paper we consider a class of rings which we shall call the class of weakly left perfect Noetherian rings. We say that a ring R is weakly left perfect Noetherian (weakly LPN for short) if $R/P(R)$ is Noetherian and $P(R)$ is left T–nilpotent. It is clear that a Noetherian or left perfect ring is weakly LPN. The converse is not necessarily true. For example the ring of integers $\mathbb{Z}$ is weakly LPN but not left perfect (or semiperfect) since $\mathbb{Z}/J(\mathbb{Z}) \cong \mathbb{Z}$ is not Artinian. In fact, we shall show in section one that the class of left perfect rings is strictly contained in the intersection of the class of semiperfect rings and the class of weakly LPN rings. The rest of this paper is concerned with necessary and sufficient conditions for a group ring to be weakly LPN. In particular, we shall show that if R is weakly LPN and G is finite, then RG is weakly LPN. We also show that the converse of this holds if R is a division ring and G is a locally finite abelian group.

1. A Proposition and Some Examples. We first prove the equivalent conditions stated in the introduction.

<u>Lemma 1.1</u>. Let I be an ideal of a ring R. If I is left T–nilpotent, then every element of I is strongly nilpotent.

<u>Proof</u>. Suppose that there exists an element $a \in I$ such that a is not strongly nilpotent. Then we have a sequence $\{x_n\}_{n\geq 0}$ where

$$x_0 = a \text{ and } x_i \in x_{i-1}Rx_{i-1}, \quad i \geq 1$$

such that $x_n \neq 0$ for every $n \geq 0$. Note that for each $i \geq 1$, x_i can be written in the form

$$x_i = (as_1)(as_2)\cdots(as_{2^i-1})a$$

for some $s_j \in R$, $j = 1, \dots, 2^i - 1$. Since $x_n \neq 0$ for every $n \geq 1$, so

$$(as_1)(as_2)\cdots(as_{2^n-1}) \neq 0$$

for every $n \geq 1$. Then since each $as_i \in I$ and I is left T–nilpotent, we have a contradiction. Therefore every element of I must be strongly nilpotent.

Lemma 1.2. For any ring R such that $J(R)$ is left T–nilpotent, $P(R) = J(R)$.

Proof. Clearly we only need to show that $J(R) \subseteq P(R)$. But this follows easily from the fact that every element of $J(R)$ is strongly nilpotent (by Lemma 1.1) and that $P(R)$ consists of all strongly nilpotent elements of R.

Lemma 1.3. For any ring R such that $R/P(R)$ is Artinian, $J(R) = P(R)$.

Proof. Since $R/P(R)$ is Artinian, so its prime and Jacobson radicals coincide. Let $\pi \colon R \to R/P(R)$ be the canonical homomorphism. Then

$$\pi(J(R)) \subseteq J(R/P(R)) = P(R/P(R)) = \{0\}.$$

Therefore, $J(R) \subseteq \operatorname{Ker} \pi = P(R)$ and consequently, $J(R) = P(R)$.

By Lemmas 1.2 and 1.3 the following result is immediate.

Proposition 1.4. The following conditions are equivalent for a ring R:

(i) R is left perfect;
(ii) $R/P(R)$ is Artinian and $P(R)$ is left T–nilpotent;
(iii) $R/P(R)$ is Artinian and $J(R)$ is left T–nilpotent.

We now show the existence of a semiperfect ring which is not weakly LPN.

Example 1.1. Let $R = \mathbb{F}[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ be a factor ring of polynomials where $\mathbb{F}$ is a field. Then R is a commutative local ring with $P(R) = J(R)$ nil, but not T–nilpotent. Hence, R is semiperfect but not weakly LPN.

For a non–commutative example of a semi–perfect ring which is not weakly LPN, we have

Example 1.2. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{F}$ a field and V a left $\mathbb{F}$–vector space with basis $\{x_n\}_{n \geq 0}$. Set

$$V_1 = \{0\} \text{ and } V_n = \sum_{i=1}^{n-1} \mathbb{F}x_i, \quad n \geq 2.$$

Let R consist of those $f \in \operatorname{End}_{\mathbb{F}}(V)$ such that for some scalar a_f

(i) $\dim_{\mathbb{F}} \mathrm{Im}(f - a_f 1_V) < \infty$;
(ii) $(f - a_f 1_V)(x_n) \in V_n, \quad n \in \mathbb{N}$.

Let $S = \{f \in R \mid a_f = 0\}$. By routine verification, R is a subring of $\mathrm{End}_{\mathbb{F}}(V)$ and S is an ideal of R. Note that for $n \geq 2$, $f(V_n) \subseteq V_{n-1}$ for any $f \in S$. It is left as an exercise to the reader to show that $S = J(R)$ and $R/J(R) \cong \mathbb{F}$. R is therefore a semiperfect ring. To show that R is not weakly LPN, let $e_n \colon V \to V$, $n \geq 1$ be the $\mathbb{F}$–linear transformation defined as follows:

$$e_n(x_m) = \begin{cases} 0 & \text{if } m \leq n \\ x_n & \text{if } m > n \end{cases}, \qquad m \in \mathbb{N}.$$

It is straightforward to show that e_n $(n \geq 1)$ is strongly nilpotent. Hence, $e_n \in P(R)$ for $n \geq 1$. Note that if $m > n$, then

$$\begin{aligned} e_1 \cdots e_n(x_m) &= e_1 \cdots e_{n-1}(x_n) \\ &= e_1 \cdots e_{n-2}(x_{n-1}) \\ &= \cdots \\ &= e_1(x_2) = x_1 \neq 0. \end{aligned}$$

Therefore $e_1 \cdots e_n \neq 0$ for any $n \geq 1$. It follows that $P(R)$ is not left T–nilpotent; hence, R is not weakly LPN.

We next give an example which shows that the class of left perfect rings is strictly contained in the intersection of the class of semiperfect rings and the class of weakly LPN rings.

Example 1.3. Let $\mathbb{F}$ be a field and $R = \mathbb{F}[[x]]$, the formal power series ring in x over $\mathbb{F}$. Then R is local and hence, $R/J(R)$ is a field. It follows that $R/J(R)$ is Artinian and the only idempotents of $R/J(R)$ are $0 + J(R)$ and $1 + J(R)$. It is clear then that R is semiperfect. Since R is a principal ideal domain so R is Noetherian. Then since $P(R) = \{0\}$ is left T–nilpotent and $R/P(R) \cong R$ is Noetherian, so R is weakly LPN. Note, however, that R is not left perfect since $J(R) = (x)$ is not left T–nilpotent.

2. Some Preliminary Results. In [4], Patterson defined an ideal I of a ring R to be strongly left T–nilpotent if, given any sequence $\{S_i\}_{i \geq 1}$ of finite subsets of I, there is an integer n, depending upon the sequence, such that $x_1 \cdots x_n = 0$ for all $x_i \in S_i$, $i = 1, \ldots, n$.

Proposition 2.1. (Patterson, [5]). An ideal I of a ring R is left T–nilpotent if and only if it is strongly left T–nilpotent.

We obtain a useful application of this result in the following proposition. First we recall that for any element $r = \sum_{g \in G} r_g g$ in a group ring RG, the support of r is defined as

$$\text{Supp } r = \{g \in G \mid r_g \neq 0\}.$$

Proposition 2.2. Let R be a ring and G a group. If I is a left T–nilpotent ideal of R, then IG is a left T–nilpotent ideal of RG.

Proof. Let $\{x_i\}_{i \geq 1}$ be a sequence of elements of IG. Then for each i,

$$x_i = a_{i_1} g_{i_1} + \cdots + a_{i_{n(i)}} g_{i_{n(i)}}$$

for some $a_{i_j} \in I$, $g_{i_j} \in G$ $(j = 1, \ldots, n(i))$. Now for each i, let

$$S_i = \{a_{i_1}, \ldots, a_{i_{n(i)}}\}.$$

We thus have a sequence $\{S_i\}_{i \geq 1}$ of finite subsets of I. Since I is strongly left T–nilpotent (by Proposition 2.1), there exists a positive integer m such that $S_1 \cdots S_m = \{0\}$. Now suppose that Supp $(x_1 \cdots x_m) \neq \emptyset$. Then there exists an element $g \in$ Supp $(x_1 \cdots x_m)$ and we note that the coefficient of g is a sum of elements of the form $b_1 \cdots b_m$ where $b_i \in S_i$, $i = 1, \ldots, m$. Then since $S_1 \cdots S_m = \{0\}$, so $b_1 \cdots b_m = 0$. It follows that the coefficient of g is zero; a contradiction. Therefore, Supp $(x_1 \cdots x_m) = \emptyset$ and hence, $x_1 \cdots x_m = 0$. It follows from this that IG is left T–nilpotent.

Lemma 2.3. Let R be a ring such that $R/P(R)$ is Noetherian. Then for any homomorphic image S of R, $S/P(S)$ is also Noetherian.

Proof. Let f be an epimorphism of R onto S and let $\pi\colon R/P(R) \to S/P(S)$ be the mapping induced by f, that is,

$$\pi(x + P(R)) = f(x) + P(S), \quad x \in R.$$

Since $f(P(R)) \subseteq P(S)$, so π is well–defined. It is easily verified that π is a ring epimorphism. Then since $R/P(R)$ is Noetherian, so is $S/P(S)$.

Proposition 2.4. Any homomorphic image of a weakly LPN ring is weakly LPN.

Proof. Let R be a weakly LPN ring and f a ring epimorphism of R onto a ring S. We wish to show that S is weakly LPN. By Lemma 2.3 we know that $S/P(S)$ is Noetherian. Thus it remains to show that $P(S)$ is left T–nilpotent.

We first note that $f(P(R))$ is a left T–nilpotent ideal of S. Indeed, let $\{s_1, s_2, \dots\}$ be a sequence of elements of $f(P(R))$. Then for each $i = 1, 2, \dots$, $s_i = f(r_i)$ for some $r_i \in P(R)$. Since $P(R)$ is left T–nilpotent, there exists a positive integer n such that $r_1 \cdots r_n = 0$. Therefore

$$s_1 \cdots s_n = f(r_1 \cdots r_n) = 0$$

and hence, $f(P(R))$ is left T–nilpotent.

Now define $\pi\colon R/P(R) \to S/f(P(R))$ as follows:

$$\pi\colon r + P(R) \mapsto f(r) + f(P(R)), \quad r \in R.$$

By routine verification π is a well–defined ring epimorphism. Then since $R/P(R)$ is Noetherian, so is $S/f(P(R))$. Therefore, $P(S/f(P(R)))$ is nilpotent; hence, left T–nilpotent. Now since $f(P(R)) \subseteq P(S)$, we have that $P(S)/f(P(R)) = P(S/f(P(R)))$. Therefore, $P(S)/f(P(R))$ is left T–nilpotent and since $f(P(R))$ is also left T–nilpotent, so is $P(S)$.

Proposition 2.5. Let R be a ring and G a locally finite group. Then RG is Noetherian if and only if R is Noetherian and G is finite.

Proof. This follows easily from [2] and the fact that a group G has the maximal condition for subgroups if and only if G and all its subgroups are finitely generated.

3. Sufficient Conditions. We first obtain sufficient conditions for a group ring to be weakly LPN.

Proposition 3.1. Let R be a ring and G a group. If R is weakly LPN and G is finite, then RG is weakly LPN.

Proof. Since R is weakly LPN, so $R/P(R)$ is Noetherian. Then since G is finite, $(R/P(R))G$ is Noetherian. Now let $\pi\colon RG/P(R)G \to RG/P(RG)$ be defined as follows:

$$\pi(\alpha + P(R)G) = \alpha + P(RG), \quad \alpha \in RG.$$

Since $P(R)G \subseteq P(RG)$ (by [2]), so π is well–defined. It is easily verified that π is a ring epimorphism. Then since $RG/P(R)G \cong (R/P(R))G$ is Noetherian, so is $RG/P(RG)$.

Next we show that $P(RG)$ is left T–nilpotent. Since $P(R)$ is left T–nilpotent, Proposition 2.2 tells us that $P(R)G$ is also left T–nilpotent. Let $\delta\colon RG \to RG/P(R)G$ canonically. Then

$$P(RG)/P(R)G = \delta(P(RG)) \subseteq P(RG/P(R)G). \tag{3.1}$$

Since $RG/P(R)G$ is Noetherian, so $P(RG/P(R)G)$ is nilpotent; hence, left T–nilpotent. It follows from (3.1) that $P(RG)/P(R)G$ is left T–nilpotent. Then since $P(R)G$ is also left T–nilpotent, so is $P(RG)$. Hence, RG is weakly LPN.

Remark 3.1. We note that G finite is not a necessary condition for RG to be weakly LPN. For example, if G is an infinite cyclic group then the integral group ring $\mathbb{Z}G$ is Noetherian and hence, weakly LPN.

4. Necessary Conditions. We first recall some standard notation. Let R be a ring and H a subgroup of a group G. Then ωH is the left ideal of RG generated by $\{1-h \mid h \in H\}$. In particular, if $H = G$, then $\triangle = \omega G$ is called the augmentation ideal of RG. We also recall that a nontrivial group G is said to be a prime group if it has no finite normal subgroup other than the trivial subgroup $\{1\}$.

Proposition 4.1. (Connell, [2]). Let R be a ring and G a prime group. Then RG is semiprime if and only if R is semiprime.

Lemma 4.2. Let R be a ring and G a prime group. Then $P(RG) = P(R)G$.

Proof. The inclusion $P(R)G \subseteq P(RG)$ follows from [2]. For the reverse inclusion, note that since G is prime and $R/P(R)$ is semiprime, Proposition 4.1 tells us that

$$P(RG/P(R)G) \cong P((R/P(R))G) = \{0\}.$$

As $P(RG)$ is the smallest ideal K such that $P(RG/K) = \{0\}$, it follows that $P(RG) \subseteq P(R)G$.

Our main result in this section is as follows.

Proposition 4.3. Let R be a ring and G a group. If RG is weakly LPN, then R is weakly LPN and either G has the maximal condition for subgroups or G has an infinite ascending chain of finite normal subgroups.

Proof. Since RG is weakly LPN and $R \cong RG/\triangle$, it follows from Proposition 2.4 that R is weakly LPN. Without loss of generality we may assume that R is semiprime; for $R/P(R)$ is semiprime and $(R/P(R))G \cong RG/P(R)G$ is weakly LPN.

Now if $P(RG) = \{0\}$, then $RG \cong RG/P(RG)$ is Noetherian. It follows that G has the maximal condition for subgroups. If $P(RG) \neq \{0\}$, then since $P(R) = \{0\}$, it follows from Proposition 4.1 that G is not a prime group. Therefore, G contains a finite normal subgroup $H_1 \neq \{1\}$.

We next consider $G_1 = G/H_1$. Since $RG_1 = R(G/H_1) \cong RG/\omega H_1$, it follows from Proposition 2.4 that RG_1 is weakly LPN. Now if $P(RG_1) = \{0\}$, then $RG_1 \cong RG_1/P(RG_1)$ is Noetherian. Therefore, $G_1 = G/H_1$ has the maximal condition for subgroups. As the finite subgroup H_1 also has the maximal condition for subgroups, so does G. If $P(RG_1) \neq \{0\}$, then since $P(R) = \{0\}$, it follows from Proposition

4.1 that G_1 is not prime. Therefore $G_1 = G/H_1$ contains a finite normal subgroup H_2/H_1 where H_2 is a normal subgroup of G with $H_2 \supset H_1$.

Now consider $G_2 = G/H_2$ and repeat the same argument as in the preceding paragraph. By continuing the same process, we see that we either have $P(RG_n) = \{0\}$ at some point n or $P(RG_n) \neq \{0\}$ for any n. If $P(RG_n) \neq \{0\}$ for any n, then it is clear that we would have an infinite ascending chain

$$H_1 \subset H_2 \subset \cdots$$

of finite normal subgroups of G. If $P(RG_n) = \{0\}$ for some n, then $RG_n \cong RG_n/P(RG_n)$ is Noetherian and therefore G_n has the maximal condition for subgroups. Then since $G_n = G/H_n$ and H_n is finite (hence, H_n has the maximal condition for subgroups), so G also has the maximal condition for subgroups.

5. Some Related Results. We note that every division ring (also known as skew field) is weakly LPN.

Proposition 5.1. Let R be a division ring with char $R = 0$ and let G be a locally finite group. Then RG is weakly LPN if and only if G is finite.

Proof. The sufficiency follows immediately from Proposition 3.1. Now suppose that RG is weakly LPN. Note that the order of every finite subgroup of G is a unit in R since char $R = 0$. Then since G is locally finite and R is (von Neumann) regular, it follows from [1] that RG is regular. Therefore $P(RG) \subseteq J(RG) = \{0\}$. Then since $RG \cong RG/P(RG)$ is Noetherian, it follows immediately from Proposition 2.5 that G is finite.

If we assume additionally that G is abelian, then the assertion in Proposition 5.1 also holds for any division ring of characteristic $p \geq 0$.

Proposition 5.2. Let R be a division ring with char $R = p \geq 0$ and let G be a locally finite abelian group. Then RG is weakly LPN if and only if G is finite.

Proof. Because of Proposition 5.1 we may assume that $p > 0$. The sufficiency follows immediately from Proposition 3.1.

Now suppose that RG is weakly LPN. We assume that G is infinite and derive a contradiction. Since G is a locally finite (hence, torsion) abelian group, we may write $G \cong G_p \times H$, where G_p is the Sylow p–subgroup of G and the order of every element of H is prime to p. Now since R is regular, H is locally finite and the order of every finite subgroup of H is a unit in R, we have that RH is regular (by [1]). Therefore $P(RH) = \{0\}$. Then since $RH \cong R(G/G_p) \cong RG/\omega G_p$, it follows from Proposition 2.4 that RH is weakly LPN. Therefore, $RH \cong RH/P(RH)$ is Noetherian and hence, by Proposition 2.5, H is finite. As G is assumed to be infinite, we must have that G_p is infinite.

Let $g \in G_p$, $g \neq 1$. Then $g^{p^n} = 1$ for some positive integer n. We then have that $(1-g)^{p^n} = 0$, that is, $1-g$ is a nilpotent element of RG. As $1-g$ belongs to the center of RG, so $1-g$ is a strongly nilpotent element and therefore, $1-g \in P(RG)$. Thus, we have shown that for any $g \in G_p$, $1-g \in P(RG)$.

Now construct a sequence $\{g_i\}_{i \geq 1}$ in G_p so that $g_1 \neq 1$ and for $n \geq 2$, g_n does not belong to the subgroup generated by $\{g_1, \ldots, g_{n-1}\}$. Thus, for any $n \geq 1$, the product $\prod_{i=1}^{n}(1-g_i)$ is never zero since the term $\prod_{i=1}^{n} g_i$ does not cancel. This contradicts the left T–nilpotence of $P(RG)$. Hence, G must be finite.

From Propositions 3.1 and 5.2 we have the following obvious corollary.

Corollary 5.3. Let R be a division ring with char $R = p \geq 0$ and let G be a locally finite abelian group. Then RG is weakly LPN if and only if RH is weakly LPN for every subgroup H of G.

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