ON FUNCTIONS WITH LINDELÖF POINT INVERSES

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In this note we improve upon the following two well–known results from general topology.

(1) If g is a closed surjection from a topological space X to a Lindelöf space Y with $g^{-1}(y)$ compact for each $y \in Y$, then X is Lindelöf (see [1]).
(2) Let X, Y be topological spaces. If X is compact and Y is Lindelöf, then the product space $X \times Y$ is Lindelöf.

The results which we establish are the following.

Theorem. If g is a closed function from a topological space X to a Lindelöf space Y with $g^{-1}(y)$ Lindelöf for each $y \in Y$, then X is Lindelöf.

Corollary. Let X, Y be Lindelöf topological spaces. If the projection $\pi: X \times Y \to Y$ is a closed function, then the product space $X \times Y$ is Lindelöf.

We use the facts that a topological space is compact (Lindelöf) if every family of closed subsets of the space with the finite (countable) intersection property has a nonempty intersection.

Proof of the Theorem. Let Ω be a family of closed subsets of X with the countable intersection property, and let Ω^* be the family of countable intersections of members of Ω. Then Ω^* is a family of closed subsets of X satisfying the countable intersection property. It follows that $\{g(F) \ : \ F \in \Omega^*\}$ is a family of closed subsets of Y satisfying the countable intersection property. Since Y is Lindelöf, there is a $v \in Y$ with $v \in \cap_{\Omega^*} g(F)$. It follows that the collection of intersections of elements of Ω with $g^{-1}(v)$ is a family of closed subsets of $g^{-1}(v)$ satisfying the countable intersection property. Thus, there is a $z \in g^{-1}(v)$ such that $z \in F$ for each $F \in \Omega$. The proof is complete.

Proof of the Corollary. For each $y \in Y$, $\pi^{-1}(y)$ is homeomorphic to X and therefore Lindelöf. The proof is completed by applying the Theorem.

Finally we remark that the Theorem and Corollary improve results (1) and (2) since every compact set is Lindelöf and for any compact space X, $\pi: X \times Y \to Y$ is closed for any space Y (see [2]).

References

1. J. Dugundji, *Topology*, Allyn & Bacon, Inc., Boston, Massachusetts, 1966.
2. N. Bourbaki, *General Topology*, Hermann, Paris, France – Addison Wesley, Redding, Massachusetts, 1966.

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