FINITE CYCLES OF CERTAIN PERIODICALLY LINEAR PERMUTATIONS

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Introduction. The $3x+1$ problem, also known as the Syracuse problem, has circulated in mathematical circles sufficiently long to generate a host of generalizations. See [1] for a historical sketch. One line of generalization concerns the cycle structure of certain permutations of the integers. By a permutation of $\mathbb{Z}$ we simply mean a bijection of $\mathbb{Z}$ with itself. In this paper we consider a subset of the permutations on $\mathbb{Z}$ that, while simple enough to describe, are fascinatingly difficult to analyze. The particular example that motivates the general discussion is the mapping from $\mathbb{N}$ into $\mathbb{N}$ defined by:

$$F(n) = \begin{cases} \frac{2}{3}n, & \text{if } n \equiv 0 \pmod 3 \\ \frac{4}{3}n - \frac{1}{3}, & \text{if } n \equiv 1 \pmod 3 \\ \frac{4}{3}n + \frac{1}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

It is relatively easy to verify that F is a permutation of $\mathbb{N}$. The inverse is given explicitly by:

$$F^{-1}(n) = \begin{cases} \frac{3}{2}n, & \text{if } n \equiv 0 \pmod 2 \\ \frac{3}{4}n + \frac{1}{4}, & \text{if } n \equiv 1 \pmod 4 \\ \frac{3}{4}n - \frac{1}{4}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(Note that any permutation F defined on $\mathbb{N}$ may be extended to a permutation of $\mathbb{Z}$ by defining $F(0) = 0$, and $F(-n) = -F(n)$ for all n in $\mathbb{N}$.)

We will refer to F as the Collatz permutation in honor of the German mathematician Lothar Collatz, who originated both the '$3x+1$ Problem' and the problem of determining the cycle structure of F while a student at the University of Heidelberg in the 1920's.

In section 1 we define the set of periodically linear permutations, of which the Collatz permutation is one example, and the subset of permutations of type V. In section 2 we state and prove the main result: given any permutation of type V and any $s \in \mathbb{N}$ the number of cycles of length s is finite. The main result is then used to deduce that the finite cycles of the Collatz permutation of length less than or equal to 50 are precisely those already known:

$$(1)$$
$$(2\ \ 3)$$
$$(4\ \ 5\ \ 7\ \ 9\ \ 6)$$
$$(44\ \ 59\ \ 79\ \ 105\ \ 70\ \ 93\ \ 62\ \ 83\ \ 111\ \ 74\ \ 99\ \ 66).$$

Finally, we discuss some conjectures suggested by the proof of the main result.

1. Finite Cycles of Permutations of Type V. For $p \in \mathbb{N}$, let $R(p) = \{n \mid 0 \leq n \leq p-1\}$.

Definition 1. A periodically linear permutation of $\mathbb{Z}$ is a permutation so that

$$g(m) = a_r m + b_r, \quad \text{if} \quad m \equiv r \pmod{p},$$

where p is a positive integer, and $a_r > 0$ and b_r are rational numbers for all $r \in R(p)$.

Writing $a_r = t_r/p$ for $t_r \in \mathbb{N}$, and setting $m = pq+r$ for $r \in R(p)$, a periodically linear permutation g may be written as

$$g(pq+r) = t_r q + g(r), \quad r \in R(p). \tag{$*$}$$

Using this notation, we state a result of G. Venturini [4].

Theorem. A function g of the form given by equation $(*)$ is a permutation of $\mathbb{Z}$ if and only if:

(i) For $i, j \in R(p)$, $i \neq j$ implies $g(i) \neq g(j)$ modulo $\gcd(t_i, t_j)$.

(ii)

$$\sum_{r=0}^{p-1} \frac{1}{t_r} = 1.$$

This result is useful in constructing examples of periodically linear permutations. In particular, one has the set of permutations given by:

Definition 2. Let $p > 1$ be an odd integer. Then the permutation g_p defined by

$$g_p(pq+r) = (p + (-1)^r)q + r, \quad r \in R(p)$$

is said to be a permutation of type V.

(Note that if f is the trivial permutation of $\mathbb{Z}$ defined by $f(n) = n - 1$, then $f \circ g_3$ is the Collatz permutation. Hence, the cycle structure of F and g_3 are identical.)

It is hoped that the permutations of type V are a class narrow enough to be successfully investigated, but broad enough to provide insight into the behavior of a general periodically linear permutation.

As discussed in [4], every cycle of g_p is either contained in $\mathbb{Z}^+$ or the negative of the cycle is contained in $\mathbb{Z}^+$. Note also that for any p, $g_p(0) = 0$; so it suffices to consider the action of g_p on $\mathbb{N}$. A permutation g_p fixes r for all $r \in R(p)$, and the cycle structure of g_p includes the $(p-1)/2$ transpositions $(p, p+1)$, $(p+2, p+3)$, $\ldots$, $(2p-3, 2p-2)$. Hence, the permutations of type V witness that a periodically linear permutation may have arbitrarily many finite cycles. The main result of this paper is that for a given permutation of type V, the number of cycles of a given length is finite.

To prove this result we will need several lemmas. We begin by fixing some notation. Writing $n = pm + r$ for $r \in R(p)$, let $\mathcal{E} = \{n \mid r \text{ is even}\}$ and let $\mathcal{O} = \{n \mid r \text{ is odd}\}$. Given a finite cycle of length s, say $(n_1, n_2, \ldots, n_s)$, we define $t = |\{n_i \mid n_i \in \mathcal{O}\}|$. The symbols s and t will keep these meanings for the remainder of the discussion. We also assume that a cycle is written so that n_1 is the smallest member of the cycle.

Our first lemma is motivated by noting:

$$g_p(n) = \begin{cases} \frac{p+1}{p}n - \frac{r}{p}, & \text{if } n \in \mathcal{E} \\ \frac{p-1}{p}n + \frac{r}{p}, & \text{if } n \in \mathcal{O}. \end{cases}$$

<u>Lemma A</u>. Given any $\delta > 0$ there is an integer N_δ so that for all $n \geq N_\delta$

$$\left(\frac{p+1}{p} - \delta\right)n < g_p(n) < \left(\frac{p+1}{p} + \delta\right)n, \text{ if } n \in \mathcal{E},$$

$$\left(\frac{p-1}{p} - \delta\right)n < g_p(n) < \left(\frac{p-1}{p} + \delta\right)n, \text{ if } n \in \mathcal{O}.$$

<u>Proof</u>. Elementary algebra.

For the sake of conciseness, define the functions

$$A(x) \equiv \frac{\log\left(\frac{p+1+xp}{p}\right)}{\log\left(\frac{p}{p-1}\right)}, \qquad B(x) \equiv \frac{\log\left(\frac{1+\frac{p}{p-1}x}{1+\frac{p}{p+1}x}\right)}{\log\left(\frac{p}{p-1}\right)}.$$

<u>Lemma B</u>. Suppose $(n_1, \ldots, n_s)$ is a finite cycle of a permutation g_p. Let $\delta > 0$ be given and let N_δ be as in Lemma A. Then if $n_1 \geq N_\delta$,

$$s(A(-\delta) + B(-\delta)) < t < s(A(\delta) + B(\delta)).$$

<u>Proof</u>. Since $(n_1, \ldots, n_s)$ is a cycle, $g_p^{(s)}(n_1) = n_1$. Applying Lemma A,

$$\left(\frac{p+1}{p} - \delta\right)^{s-t} \left(\frac{p-1}{p} - \delta\right)^t n_1 < n_1 < \left(\frac{p+1}{p} + \delta\right)^{s-t} \left(\frac{p-1}{p} + \delta\right)^t n_1.$$

After some algebraic manipulation one has:

$$s(A(-\delta) + tB(-\delta)) < t < s(A(\delta) + tB(\delta)).$$

The function $B(x)$ has the same sign as x, and by definition $t \leq s$, hence, the desired inequality follows.

<u>Lemma C</u>. Let $\gamma_p = \log((p+1)/p)/\log(p/(p-1))$. For all integers $p > 1$, γ_p is irrational.

<u>Proof</u>. Suppose, by way of contradiction, that there exist $b, c \in \mathbb{N}$ so that $\gamma_p = b/c$. Then $p^{b+c} = (p-1)^b(p+1)^c$. Since $p > 1$, $p+1$ has at least one prime factor that is not a factor of p; but this contradicts the unique factorization of the integers.

Now for the main result.

<u>Theorem 1</u>. Let $p > 1$ be an odd integer and let g_p be a permutation of type V. For any $s \in \mathbb{N}$, the number of cycles of length s is finite.

<u>Proof</u>. Suppose, by way of contradiction, there is an s so that there are infinitely many cycles of length s. As before, if the cycle of length s is written as $(n_1, \ldots, n_s)$, one may assume without loss of generality that n_1 is the smallest of

the n_i. Since for $1 \le i \le s$, $n_i < (2^s+1)n_1$, it follows that there are cycles of length s for which n_1 is arbitrarily large.

Note that $\lim_{x\to 0} A(x) = \gamma_p$ and $\lim_{x\to 0} B(x) = 0$. By Lemma C, γ_p is irrational, hence also $s\gamma_p$ is irrational. Since $s\gamma_p$ is irrational, one may find a $\delta > 0$ so that the interval $[s(A(-\delta)+B(-\delta)), s(A(\delta)+B(\delta))]$ does not contain an integer. For such a δ choose an integer N_δ as described in Lemma A. Choose a cycle of length s so that $n_1 > N_\delta$. By Lemma B the interval $[s(A(-\delta)+B(-\delta)), s(A(\delta)+B(\delta))]$ contains the integer t, a contradiction.

The ideas contained in the proof of Theorem 1 may be used to find all cycles of a given permutation g_p not exceeding a given length. For a given $s \in \mathbb{N}$, one may compute a $\delta > 0$ small enough so that the interval $[s(A(-\delta)+B(-\delta)), s(A(\delta)+B(\delta))]$ does not contain an integer. Using this value of δ, one may compute the corresponding N_δ of Lemma A. Now given a cycle of length s there is a real number B so that $n_i < Bn_1$ for $1 \le i \le s$. The value $B = 2^s+1$ works, but for efficient computation one wants the smallest possible B. It follows that all cycles of length not exceeding s are contained in the interval $[1, BN_\delta]$. This reduces the search for cycles of length s to a finite interval. Doing the necessary checking for $p = 3$ (with the help of a computer), one has the following corollary to Theorem 1.

Corollary. The only finite cycles of the Collatz permutation of length not exceeding 50 are the four listed in the introduction.

2. A Conjecture on the Cycle Lengths for Permutations of Type V. The first four positive convergents of the continued fraction expansion of γ_3 are 1/1, 1/2, 3/5, and 7/12—which are precisely the ratios t/s for the four known finite cycles of g_3! In [3] D. Shanks gives a heuristic argument that the denominators $1, 2, 5, 12, \ldots$ of the continued fraction expansion of γ_3 are the 'most likely' cycle lengths for the Collatz permutation. The proof of Theorem 1 puts these arguments in sharper focus. The ratio t/s must closely approximate γ_p when n_1 is large, and the convergents t_i/s_i of γ_p are the 'best possible' approximations to γ_p, in the sense made precise by the well–known theorem [4]:

Theorem. For any $a, b \in \mathbb{N}$ with $1 \le b \le s_i$, $|t_i/s_i - \gamma_p| \le |a/b - \gamma_p|$.

For convenience let $\Lambda(p)$ be the set of all positive integers that are the denominator of some convergent of γ_p. So, for example, one computes

$$\begin{aligned}
\Lambda(3) &= \{1, 2, 5, 12, 41, 53, 306, \ldots\} \\
\Lambda(5) &= \{1, 2, 9, 20, 129, 149, 1619, \ldots\} \\
\Lambda(7) &= \{1, 2, 13, 28, 265, 293, \ldots\}.
\end{aligned}$$

By the corollary to Theorem 1, g_3 has no cycle of length 41, hence, $s \in \Lambda(p)$ is not a sufficient condition for g_p to have a cycle of length s. On the other hand, g_5 has a cycle of length 7 (with $n_1 = 9$) and g_7 has a cycle of length 11 (with $n_1 = 25$), so $s \in \Lambda(p)$ is not a necessary condition for γ_p to have a cycle of length s. However, the arguments embodied in the proof of Theorem 1 and the numerical evidence, both suggest that all but finitely many of the cycle lengths are elements of $\Lambda(p)$.

Conjecture. If g_p is a permutation of type V and s is the length of a finite cycle of g_p, then there is an integer N_p so that $s \geq N_p$ implies $s \in \Lambda(p)$.

The finite cycles of g_p can occur quite unexpectedly: g_7 has a cycle of length 265 with $n_1 = 1621$, and a cycle of length 293 with $n_1 = 3313$. Hence, there are reasonable grounds for doubting the conjecture that the number of finite cycles of a periodically linear permutation of type V are finite. A proof or counterexample to the conjecture given above would be a useful intermediate step.

References

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