SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

117. [1998, 117] *Proposed by Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.*

(a) Prove that

$$\int_0^\infty \frac{1}{1+x^3}\cdot\frac{1}{1+r^3x^3}\cdots\frac{1}{1+(r^nx)^3}dx$$

$$= \frac{2\pi}{3\sqrt{3}}\cdot\frac{(1-r^2)(1-r^5)\cdots(1-r^{3n-1})}{(1-r^3)(1-r^6)\cdots(1-r^{3n})}.$$

(b) Prove that

$$\int_0^\infty \frac{1}{1+x^4}\cdot\frac{1}{1+r^4x^4}\cdots\frac{1}{1+(r^nx)^4}dx$$

$$= \frac{\pi}{2\sqrt{2}}\cdot\frac{(1-r^3)(1-r^7)\cdots(1-r^{4n-1})}{(1-r^4)(1-r^8)\cdots(1-r^{4n})}.$$

Solution by Larry Eifler and Jerry Masuda, University of Missouri–Kansas City, Missouri.

Fix $m>1$ and let $r>0$. For $n=0,1,2,\ldots$ and for $x\geq 0$, define

$$\phi_n(r,x) = \prod_{k=0}^{n}\frac{1}{1+(r^kx)^m}$$

$$= \frac{1}{1+x^m}\cdot\frac{1}{1+(rx)^m}\cdots\frac{1}{(r^nx)^m}.$$

Set

$$I_n(r) = \int_0^\infty \phi_n(r,x)dx \qquad \text{for } n=0,1,2,\ldots.$$

It is well–known that (see Murray Spiegel, *Complex Variables*, McGraw–Hill, 1964, p. 185)

$$\int_0^\infty \frac{y^p}{1+y}\frac{dy}{y} = \frac{\pi}{\sin(\pi p)} \qquad \text{for } 0 < p < 1$$

and so

$$I_0(r) = \int_0^\infty \frac{1}{1+x^m}dx = \frac{1}{m}\int_0^\infty \frac{y^{1/m}}{1+y}\frac{dy}{y} = \frac{\pi/m}{\sin(\pi/m)}.$$

For $n = 1, 2, \ldots$, we have

$$(1+x^m)\phi_n(r,x) = \phi_{n-1}(r,rx)$$

and

$$(1+(r^n x)^m)\phi_n(r,x) = \phi_{n-1}(r,x).$$

Thus,

$$\begin{aligned} I_{n-1}(r) &= \int_0^\infty \phi_{n-1}(r,x)dx = \int_0^\infty \phi_{n-1}(r,rx)rdx \\ &= \int_0^\infty r(1+x^m)\phi_n(r,x)dx \qquad (1) \end{aligned}$$

and

$$I_{n-1}(r) = \int_0^\infty (1+(r^{mn}x^m))\phi_n(r,x)dx. \qquad (2)$$

Using (1) and (2), we find

$$\begin{aligned} (1-r^{mn-1})I_{n-1}(r) &= \int_0^\infty [(1+(r^{mn}x^m) - r^{mn}(1+x^m)]\phi_n(r,x)dx \\ &= (1-r^{mn})I_n(r). \end{aligned}$$

It follows that

$$I_n(r) = \frac{\pi/m}{\sin(\pi/m)} \prod_{k=1}^{n} \left(\frac{1 - r^{mk-1}}{1 - r^{mk}} \right) \quad \text{for } r \neq 1.$$

Parts (a) and (b) of the problem are special cases of the above formula. Taking the limit as r approaches 1, we find that

$$\int_0^\infty \left(\frac{1}{1 + x^m} \right)^n dx = \frac{\pi/m}{\sin(\pi/m)} \prod_{k=1}^{n} \left(1 - \frac{1}{mk} \right).$$

Also solved by the proposer.

118. [1998, 118] *Proposed by Xuming Chen, 1126 Casson Green NW, Calgary, Alberta, Canada and Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.*

Let

$$S(a, k, n) = \sum_{i=1}^{n} a^{i-1} i^k.$$

Show that

$$(1 - a)S(a, k, n) + \sum_{m=1}^{k} \binom{k}{m} (-1)^m S(a, k - m, n) = -a^n n^k.$$

Solution I by the proposers.

$$
\begin{aligned}
(1-a)S(a,k,n) &= \sum_{i=1}^{n} a^{i-1} i^k - \sum_{i=1}^{n} a^i i^k \\
&= \sum_{i=1}^{n} a^{i-1} i^k - \sum_{j=2}^{n+1} a^{j-1} (j-1)^k \\
&= \sum_{i=1}^{n} a^{i-1} i^k - \sum_{j=1}^{n+1} a^{j-1} (j-1)^k \\
&= \sum_{i=1}^{n} a^{i-1} i^k - \sum_{j=1}^{n} a^{j-1} (j-1)^k - a^n n^k \\
&= \sum_{i=1}^{n} a^{i-1} i^k - \sum_{i=1}^{n} a^{i-1} (i-1)^k - a^n n^k \\
&= \sum_{i=1}^{n} a^{i-1} \left[i^k - (i-1)^k \right] a^{i-1} - a^n n^k \\
&= \sum_{i=1}^{n} \left[i^k - \sum_{m=0}^{k} \binom{k}{m} i^{k-m} (-1)^m \right] a^{i-1} - a^n n^k \\
&= \sum_{i=1}^{n} \left[i^k - \left(i^k + \sum_{m=1}^{k} \binom{k}{m} i^{k-m} (-1)^m \right) \right] a^{i-1} - a^n n^k \\
&= \sum_{i=1}^{n} - \left(\sum_{m=1}^{k} \binom{k}{m} i^{k-m} (-1)^m \right) a^{i-1} - a^n n^k \\
&= - \sum_{m=1}^{k} \binom{k}{m} (-1)^m \left[\sum_{i=1}^{n} i^{k-m} a^{i-1} \right] - a^n n^k \\
&= - \sum_{m=1}^{k} \binom{k}{m} (-1)^m S(a, k-m, n) - a^n n^k.
\end{aligned}
$$

Solution II by Jose Luis Diaz, Universitat Politecnica de Catalunya, Colom 1, Terrassa, Spain.

We observe that

$$P(a) = (1-a)S(a,k,n) + \sum_{m=1}^{k} \binom{k}{m} (-1)^m S(a,k-m,n)$$

is a polynomial of degree n in the variable a. Hence,

$$\begin{aligned}
P(a) = (1-a)S(a,k,n) &+ \binom{k}{1}(-1)^1 S(a,k-1,n) \\
&+ \binom{k}{2}(-1)^2 S(a,k-2,n) + \cdots + \binom{k}{k}(-1)^k S(a,0,n) \\
= &\binom{k}{0}(-1)^0 \left[1^k + 2^k a + \cdots + n^k a^{n-1}\right] \\
&+ \binom{k}{1}(-1)^1 \left[1^{k-1} + 2^{k-1} a + \cdots + n^{k-1} a^{n-1}\right] \\
&+ \binom{k}{2}(-1)^2 \left[1^{k-2} + 2^{k-2} a + \cdots + n^{k-2} a^{n-1}\right] + \cdots \\
&+ \binom{k}{k}(-1)^k \left[1^0 + 2^0 a + \cdots + n^0 a^{n-1}\right] - \left[1^k a + 2^k a^2 + \cdots + n^k a^n\right].
\end{aligned}$$

Now ordering the terms of $P(a)$ in increasing powers of a we get the result. In fact,

$$P(a) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j + \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j 2^{k-j} - 1^k\right] a$$

$$+ \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j 3^{k-j} - 2^k\right] a^2 + \cdots + \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j n^{k-j} - n^k\right] a^{n-1} - n^k a^n$$

$$= (1-1)^k + [(2-1)^k - 1^k]a + [(3-1)^k - 2^k]a^2 + \cdots$$

$$+ [(n-1)^k - (n-1)^k]a^{n-1} - n^k a^n = -a^n n^k.$$

Solution III by Carl Libis, 13126 Courbet Lane, Granada Hills, California.

First note that

$$S(a, j+1, n) = \frac{d}{da}\Big[aS(a,j,n)\Big] = S(a,j,n) + a\frac{d}{da}\Big[S(a,j,n)\Big]$$

for $j = 0, 1, 2, \ldots$. Let $S(a,*,n)^j := S(a,j,n)$. Now note that

$$(1-a)S(a,k,n) + \sum_{m=1}^{k} \binom{k}{m} (-1)^m S(a, k-m, n) = -a^n n^k, \qquad k = 1, 2, 3, \ldots \quad (1)$$

is equivalent to

$$\sum_{m=0}^{k} \binom{k}{m} (-1)^m S(a, k-m, n) = aS(a,k,n) - a^n n^k = aS(a,k,n-1), \quad k = 1, 2, 3, \ldots$$

which is equivalent to

$$(S(a,*,n) - 1)^k = aS(a,k,n-1), \quad k = 1, 2, 3, \ldots. \qquad (2)$$

To prove (1) we will prove (2) by induction. When $k=1$, then

$$(S(a,*,n)-1)^1 = S(a,1,n)-S(a,0,n) = \sum_{i=1}^{n} a^{i-1}(i-1) = \sum_{i=2}^{n} a^{i-1}(i-1)$$

$$= \sum_{i=1}^{n-1} a^i i = a\sum_{i=1}^{n-1} a^{i-1} i = aS(a,1,n-1).$$

Assume that $(S(a,*,n)-1)^k = aS(a,k,n-1)$. Then

$$(S(a,*,n)-1)^{k+1} = \sum_{m=0}^{k+1} \binom{k+1}{m}(-1)^m S(a,k+1-m,n)$$

$$= \sum_{m=0}^{k+1}\left[\binom{k}{m-1}+\binom{k}{m}\right](-1)^m S(a,k+1-m,n)$$

$$= \sum_{m=0}^{k+1}\binom{k}{m-1}(-1)^m S(a,k+1-m,n) + \sum_{m=0}^{k+1}\binom{k}{m}(-1)^m S(a,k+1-m,n)$$

$$= \sum_{m=1}^{k+1}\binom{k}{m-1}(-1)^m S(a,k+1-m,n) + \sum_{m=0}^{k}\binom{k}{m}(-1)^m S(a,k+1-m,n)$$

$$= \sum_{m=0}^{k}\binom{k}{m}(-1)^{m+1} S(a,k-m,n)$$

$$+ \sum_{m=0}^{k}\binom{k}{m}(-1)^m \left[S(a,k-m,n) + a\frac{d}{da}S(a,k-m,n)\right]$$

$$= a\sum_{m=0}^{k}\binom{k}{m}(-1)^m \frac{d}{da}\sum_{i=1}^{n} a^{i-1} i^{k-m}$$

$$= a\sum_{m=0}^{k}\binom{k}{m}(-1)^m\sum_{i=1}^{n}(i-1)a^{i-2}i^{k-m}$$

$$= a\sum_{m=0}^{k}\binom{k}{m}(-1)^m\sum_{i=2}^{n}(i-1)a^{i-2}i^{k-m}$$

$$= a\sum_{m=0}^{k}\binom{k}{m}(-1)^m\sum_{i=1}^{n-1}ia^{i-1}(i+1)^{k-m}$$

$$= a\sum_{i=1}^{n-1}ia^{i-1}\sum_{m=0}^{k}\binom{k}{m}(-1)^m(i+1)^{k-m}$$

$$= a\sum_{i=1}^{n-1}ia^{i-1}(i+1-1)^k$$

$$= a\sum_{i=1}^{n-1}a^{i-1}i^{k+1}$$

$$= aS(a,k+1,n-1).$$

This completes the proof by induction of (2) and thus proves (1).

119. [1998, 118] *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.*

If x and y are distinct positive real numbers, prove that

$$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{5(x+y)^3}{5(x+y)^2+(x-y)^2} < \frac{x-y}{\sinh\left(\frac{x-y}{x+y}\right)}.$$

Solution I by the proposers.

On $[0,1]$, $\sinh\theta < \theta + \theta^3/5$. This can be easily proven by showing that $f(\theta) = \sinh\theta - \theta - \theta^3/5$ is decreasing on $[0,1]$ and $f(0) = 0$. Without loss of generality, assume that $x > y$. So, since $0 < (x-y)/(x+y) < 1$,

$$\sinh\left(\frac{x-y}{x+y}\right) < \frac{x-y}{x+y} + \frac{1}{5}\left(\frac{x-y}{x+y}\right)^3.$$

It is straightforward to show that this inequality is equivalent to

$$\frac{x-y}{\sinh\left(\frac{x-y}{x+y}\right)} > \frac{5(x+y)^3}{5(x+y)^2 + (x-y)^2} = \frac{5x^3 + 15x^2y + 15xy^2 + 5y^3}{6x^2 + 6y^2 + 8xy}.$$

The desired lower bound for the proposed inequality follows if it can be shown that

$$\frac{5x^3 + 15x^2y + 15xy^2 + 5y^3}{6x^2 + 6y^2 + 8xy} > \frac{(\sqrt{x} + \sqrt{y})^2}{2}$$

which is equivalent to showing that

$$x^3 + y^3 + 4x^2y + 4xy^2 > \sqrt{xy}(3x^2 + 3y^2 + 4xy).$$

Squaring both sides of the latter inequality, simplifying and factoring gives $(x - y)(x^5 - y^5) > 0$ – which is true since $x > y$ and the desired inequalities follow.

Solution II by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.

We use the method in Problem 567 of the *College Mathematics Journal*, Volume 28, Number 1 (1997), page 69. Let

$$z = \frac{x-y}{x+y} \quad \text{and} \quad \sqrt{1-z^2} = \frac{2\sqrt{xy}}{x+y}.$$

Then the above inequality becomes

$$\frac{1+\sqrt{1-z^2}}{2} < \frac{5}{5+z^2} < \frac{z}{\sinh z}$$

or inverting

$$\frac{2(1-\sqrt{1-z^2})}{z^2} > 1+\frac{z^2}{5} > \frac{\sinh z}{z} = 1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots .$$

Now

$$\frac{2(1-\sqrt{1-z^2})}{z^2} = 1+\frac{z^2}{4}+\frac{1\cdot 3}{3!2}z^4+\cdots+\frac{1\cdot 3\cdots(2n-3)}{n!2^{n-1}}z^{2n-2}+\cdots$$

so the first inequality follows. For the 2nd inequality, we show

$$f(z) = z+\frac{z^3}{5}-\sinh z > 0 \quad \text{for} \quad 0<z<1,$$

$$f(0)=0,$$

$$f'(z) = 1+\frac{3z^2}{5}-\cosh z \quad \text{and} \quad f'(0)=0$$

$$f''(z) = \frac{6z}{5}-\sinh z \quad \text{and} \quad f''(0)=0$$

$$f'''(z) = \frac{6}{5}-\cosh z \quad \text{and} \quad f'''(0)=\frac{1}{5}>0.$$

Hence, $f(z)>0$ and the 2nd inequality follows.

Also solved by N. J. Kuenzi and Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; and Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

120. [1998, 118] *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, Tennessee.*

Let a, b, and c be the sides of a right triangle such that h is the altitude to hypotenuse c. Prove that $c - 2h < a + b < 2c - h$.

Solution I by Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Joseph Wiener, University of Texas–Pan American, Edinburg, Texas; and Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

It is well known that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Hence, $c < a + b$ so

$$c - 2h < a + b. \tag{$*$}$$

By finding the area of the given triangle, we get $(ab)/2 = (ch)/2$. Since $ab = ch$ and $c^2 = a^2 + b^2$,

$$\begin{aligned}(2c - h)^2 &= 4c^2 - 4ch + h^2 > 4(c^2 - ch)\\ &= 4(a^2 + b^2 - ab) = (a + b)^2 + 3(a - b)^2 \geq (a + b)^2.\end{aligned}$$

This implies that

$$2c - h > a + b. \tag{$**$}$$

By $(*)$ and $(**)$, we get the required inequality

$$c - 2h < a + b < 2c - h.$$

Solution II by Hushang Poorkarimi and Donald Skow; University of Texas–Pan American, Edinburg, Texas.

Let $c = x + y$, where x is the projection of b to side c and y is the projection of a to side c. Then, $b + h > x$ and $a + h > y$. Therefore, $a + b + 2h > x + y$ so $a + b + 2h > c$. Hence,

$$a + b > c - 2h. \tag{1}$$

It is known that the altitude h is less than the median to side c, that is, $h < c/2$, and that $a + b \leq \sqrt{2}c$ which implies that $a + b < (3c)/2$. Thus, $a + b < (3c)/2$ and $h < c/2$ implies $a + b + h < 2c$. Hence,

$$a + b < 2c - h. \tag{2}$$

Inequalities (1) and (2) imply $c - 2h < a + b < 2c - h$.

Also solved by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico and the proposer.