A "DOUBLE" CAUCHY–SCHWARZ TYPE INEQUALITY

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Abstract. A "double" version of the C–S inequality in any complex pre–Hilbert space is given, along with some numerical applications.

A substantial part of the mathematical folklore in the frame of inner–product spaces involves applications and/or extensions of the Cauchy–Schwarz (C–S) inequality

$$|\langle e, f\rangle| \leq \|e\| \ \|f\|,$$

where by $\|\cdot\|$ we indicate the norm induced by the inner–product $\langle , \rangle$.

For the case of a (real) Hilbert space M. Lambrou (Univ. of Crete) indicated to the second author, by personal communication, the following "double" version of the C–S inequality.

$$|\langle e, f\rangle| \ |\langle e, g\rangle| \leq \frac{1}{2}\{\|f\| \ \|g\| + |\langle f, g\rangle|\}\|e\|^2. \tag{$*$}$$

(Note that if f, g are considered to be linearly dependent we simply obtain the C–S inequality.)

The fact that the R.H.S. of $(*)$ provides a better bound than the "natural" $\|e\|^2 \ \|f\| \ \|g\|$ is evident by use of the C–S inequality itself. That $(*)$ gives, in certain cases, a much better bound for $|\langle e, f\rangle||\langle e, g\rangle|$ becomes clear from the following example.

Let $f = \sin x$, $g = \cos x$, $e = 1/x$ be considered as members of the classical Hilbert space $L^2[\alpha, \beta]$, $\alpha > 0$. Then

$$|\langle e, f\rangle| \ |\langle e, g\rangle| \leq \left(\int_\alpha^\beta \frac{dx}{x^2}\right)\left(\int_\alpha^\beta \sin^2 x dx\right)^{1/2}\left(\int_\alpha^\beta \cos^2 x dx\right)^{1/2}.$$

On the other hand $(*)$ provides the bound

$$\frac{1}{2}\left(\int_\alpha^\beta \frac{dx}{x^2}\right)\left[\left(\int_\alpha^\beta \sin^2 x dx\right)^{1/2}\left(\int_\alpha^\beta \cos^2 x dx\right)^{1/2} + \left|\int_\alpha^\beta \sin x \cos x dx\right|\right].$$

Subtracting the $(*)$–bound from the C–S–bound one has

$$\frac{1}{8}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)[(\beta-\alpha+\sin 2\alpha-\sin 2\beta)^{1/2}(\beta-\alpha+\sin 2\beta-\sin 2\alpha)^{1/2}$$

$$-|\cos 2\beta-\cos 2\alpha|],$$

which increases to $+\infty$ for a (number of) suitable limit behavior of α or β.

In the present work we present an elementary proof of $(*)$ for any pre–Hilbert space over the complex field (which naturally also covers the real case), along with a few applications of $(*)$.

Theorem 1. Let e, f, g be elements of a complex pre–Hilbert space $(H, \langle , \rangle)$; then

$$2|\langle e,f\rangle|\;|\langle e,g\rangle| \le \{\|f\|\;\|g\| + |\langle f,g\rangle|\}\|e\|^2.$$

Proof. Based on a previous remark, let f, g be linearly independent, and let $e = \lambda f + \mu g$. W.L.O.G. we may also assume that $\|e\| = \|f\| = \|g\| = 1$. If $f \perp g$ the L.H.S. of $(*)$ becomes $2|\lambda|\;|\mu|$ whereas the R.H.S. becomes $|\lambda|^2 + |\mu|^2$ and we are done. If f and g are not orthogonal to each other, by the Gram–Schmidt construction we obtain $e = kf + sh$ with $h = (g - cf)(1-|c|^2)^{-1/2}$ with $c = \langle g, f\rangle \neq 0, 1$. Then $h \perp f$ and $\|h\| = 1$. It is easily seen that we may consider $c > 0$ since, otherwise, by switching from f to $(cf)/|c|$ we find ourselves in an equivalent position. Then $|\kappa|^2 + |s|^2 = 1$ with $\kappa = \langle e, f\rangle$ and $s = \langle e, h\rangle$. In a similar manner we may assume that $\kappa \ge 0$. Then

$$|\langle e,f\rangle|\;|\langle e,g\rangle| = \kappa|\kappa c + s(1-c^2)^{1/2}|.$$

But

$$|\kappa c + s(1-c^2)^{1/2}|^2 = \kappa^2c^2 + 2\mathrm{Re}(\kappa c(1-c^2)^{1/2}s) + |s|^2(1-c^2)$$

$$\le (\kappa c + |s|(1-c^2)^{1/2})^2.$$

Thus,

$$|\langle e,f\rangle|\;|\langle e,g\rangle| \le \kappa^2 c + \kappa(1-\kappa^2)^{1/2}(1-c^2)^{1/2}. \qquad (**)$$

Using the first derivative criterion, etc. for local extremes, it can be easily seen that the R.H.S. of $(**)$ is bounded by $(1+c)/2$ and we are done.

It remains now to prove $(*)$ for the case $e \notin \mathrm{sp}\{f, g\}$. Then, $e = e_1 + e_2$ with $e_1 \in \mathrm{sp}\{f, g\}$ and $e_2 \perp \mathrm{sp}\{f, g\}$. The L.H.S. of $(*)$ becomes

$$2|\langle e, f\rangle|\ |\langle e, g\rangle| \le \{\|f\|\ \|g\| + |\langle f, g\rangle|\}\|e_1\|^2,$$

because of the first part. Since $\|e_1\| \le \|e\|$ we obtain the announced result. Q.E.D.

In case $(H, \langle\cdot\rangle)$ is Hilbert, we can generalize $(*)$ as follows.

Theorem 2. For any projection P and any vectors f, g

$$2|\langle Pf, g\rangle| \equiv 2|\langle Pf, Pg\rangle| \le \|f\|\ \|g\| + |\langle f, g\rangle|.$$

Proof. Let $Q = I - P$. Then

$$\|f\|^2 = \|Pf\|^2 + \|Qf\|^2, \quad \|g\|^2 = \|Pg\|^2 + \|Qg\|^2.$$

Since

$$\langle f, g\rangle = \langle Pf, Pg\rangle + \langle Qf, Qg\rangle,$$

by the classical Schwarz inequality we have

$$|\langle f, g\rangle| \ge |\langle Pf, Pg\rangle| - \|Qf\| \cdot \|Qg\|.$$

Therefore, for the assertion, it suffices to prove that

$$|\langle Pf, Pg\rangle| + \|Qf\| \cdot \|Qg\| \le \sqrt{(\|Pf\|^2 + \|Qf\|^2)(\|Pg\|^2 + \|Qg\|^2)}.$$

Then, using the classical Schwarz inequality once more, it suffices to prove

$$\|Pf\| \cdot \|Pg\| + \|Qf\| \cdot \|Qg\| \le \sqrt{(\|Pf\|^2 + \|Qf\|^2)(\|Pg\|^2 + \|Qg\|^2)}$$

which is nothing but the classical Cauchy inequality.

Remark. The result of $(*)$ in the case where $(H, \langle\cdot\rangle)$ of Theorem 1 is considered complete corresponds to the case where $\mathrm{rank}(P) = 1$.

We turn now to a couple of applications of Theorem 1 starting with a complex Hilbert space $(H, \langle,\rangle)$.

(i) Notice first that if f and g in H are mutually orthogonal then $(*)$ reduces to

$$2|\langle e, f\rangle|\ |\langle e, g\rangle| \le \|e\|^2\ \|f\|\ \|g\| \qquad (***)$$

If now $e \in H$ and $\{f_i\}$, $i \in I$, an orthogonal family in H, we can consider finite products of the Fourier coefficients of e w.r.t. $\{f_i\}$ namely

$$\prod_{j\in J} |\langle e, f_j\rangle|,$$

where $J \subset I$ a finite set with at least two elements (in which case, naturally, we also impose $\dim H \geq 2$). It is evident that the C–S inequality would have provided the crude (upper) bound $\|e\|^n$, where n is the cardinality of J.

In view of $(***)$ though, we obtain the following far better bound, namely:

$$\prod_{j\in J} |\langle e, f_j\rangle| \leq \begin{cases} 2^{-n/2}\|e\|^n, & \text{if n is even;} \\ 2^{-(n-1)/2}\|e\|^n, & \text{if n is odd.} \end{cases}$$

(ii) For another application of $(*)$ let us employ the pre–Hilbert space $L^1(0, +\infty)$ and consider its (linearly independent) elements

$$\frac{\sin x}{x}, \quad \frac{\cos 2x}{1+x^2} \quad \text{and} \quad \frac{1}{1+x^2}$$

in the roles of e, f and g, respectively. The direct calculation of $I = \langle e, f\rangle$ is a rather painful experience within the techniques of contour integration, or even by tracing it in suitable tables.

On the other hand by elementary contour integration and/or by reference to [1], we have

$$\langle e, g\rangle = \frac{\pi}{2}(1 - e^{-1}), \quad \|f\| = \left[\frac{\pi}{8}(1 + 5e^{-4})\right]^{1/2}, \quad \|g\| = \frac{\pi^{1/2}}{2},$$

$$\langle f, g\rangle = \frac{3}{4}\pi e^{-2}, \quad \text{and} \quad \|e\|^2 = \frac{\pi}{2}.$$

Thanks to $(*)$ we obtain the following (strict) upper bound

$$I < \frac{\pi}{8(1 - e^{-1})}\left[\left(\frac{1}{2} + \frac{5}{2}e^{-4}\right)^{1/2} + 3e^{-2}\right] < 0.7112.$$

Reference

1. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, ed. A. Jeffrey, Academic Press, 1980.

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