TOTAL CHARACTERS OF DIHEDRAL GROUPS AND SHARPNESS

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Abstract. We define the total character τ of a finite group G as the sum of all its irreducible characters. A question of K. W. Johnson asks whether the total character of a finite group can be expressed as a polynomial with integer coefficients in some irreducible character χ of G. We show that in the case of dihedral groups of twice odd order the question has an affirmative answer and we give the explicit polynomial.

1. Sharp Characters. It is a well known result of Burnside (Theorem 4.3 in [3]) that if χ is a faithful character of a finite group G that takes on k distinct values, then every irreducible character of G appears as a constituent of at least one of χ, χ^2, ..., χ^{k-1}. A strengthening of this result is obtained in a modern context by Cameron and Kiyota as follows. Let G be a finite group, χ be a generalized character of G of degree n, and $L = \{\chi(g) \mid g \neq 1\}$. With this notation, Cameron and Kiyota proved in [1] in Theorem 1.1 that if $f_L(x) = \prod_{l \in L}(x - l)$, then $f_L(x) \in \mathbb{Z}[x]$ and $|G|$ divides $f_L(n)$. In the special case that $f_L(n) = |G|$, the character χ is said to be sharp. Another way to characterize a sharp character is to notice that $f_L(\chi) = \prod_{l \in L}(\chi - l1_G) = \rho$, where ρ is the character afforded by the right regular representation of G and 1_G is the trivial character of G. In other words, every irreducible character of G appears as a constituent of $f_L(\chi)$. In the same spirit, we define the total character τ of G to be the sum of all irreducible characters of G:

$$\tau = \sum_{\chi \in \mathrm{Irr}(G)} \chi.$$

A natural question that was raised first by Kenneth W. Johnson in private correspondence [5] is the following:

<u>Question 1.1</u>. For a finite group G does there necessarily exist an irreducible character χ and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$, where τ is the total character of G?

In this paper we show that if χ is any faithful irreducible character of a dihedral group of twice odd order, then Question 1.1 has an affirmative answer. In particular we prove the following theorem.

<u>Theorem A</u>. Let G be a dihedral group of order $2n$ with n odd. If χ is any faithful irreducible character of G, then a monic polynomial $g(x) \in \mathbb{Z}[x]$ exists such

that $g(\chi) = \tau$, where τ is the total character of G. Furthermore $g(x)$ is minimal with this property.

In section 2 we compute the total characters of all dihedral groups. In section 3, we prove first in Theorem 3.2 that all faithful irreducible characters of D_{2n} with n odd are sharp and then we proceed to prove our main Theorem in 3.7. It turns out that if $n \equiv 1$ or $3 \pmod 8$ then $g(x)$ has degree $(n+1)/2$ but if $n \equiv 5$ or $7 \pmod 8$ then we can get a polynomial of degree $(n-1)/2$.

Example 1.2. Recall the character table for S_3:

g_i / h_i	(1) / 1	(12) / 3	(123) / 2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
τ	4	0	1

where g_i denotes a representative of the i–th conjugacy class and h_i is the size of the i–th conjugacy class.

Notice that χ_3 is sharp since $L = \{0, -1\}$, $f_L(x) = x(x+1)$, and $f_L(2) = 2(2+1) = 6 = |S_3|$. Clearly, $\chi_3^2 = \tau$. Hence, $g(x) = x^2$.

Example 1.3. The complete character table of S_4 is:

g_i / h_i	(1) / 1	(12) / 6	(123) / 8	(12)(34) / 3	(1234) / 6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1
τ	10	0	1	2	0

Observe that both χ_4 and χ_5 are sharp with $L = \{-1, 0, 1\}$. Assume that $g(x)$ exists such that $g(\chi_4) = \tau$. But that would mean that $g(\chi_4)((12)(34)) = g(-1) = 2$ and $g(\chi_4)(1234) = g(-1) = 0$. No polynomial will satisfy that condition. Similarly for χ_5. In fact, Johnson's question has a negative answer for any irreducible character of S_4 as one easily checks.

2. Total Characters of Dihedral Groups. Let G be the dihedral group D_{2n} of order $2n$ with $n \geq 3$, so that

$$G = \langle a, b \ : \ a^n = b^2 = 1, \ b^{-1}ab = a^{-1} \rangle.$$

Using the notation in [4] we consider the character table of G for n odd and n even. We use g_i and h_i to denote a representative and the size of the i–th conjugacy class respectively.

Case 1. n odd.

The conjugacy classes of D_{2n} (n odd) are:

$$\{1\}, \ \{a^r, a^{-r}\} \ (1 \leq r \leq (n-1)/2), \ \{a^s b \mid 0 \leq s \leq n-1\}.$$

The character table of D_{2n} (n odd) where $\epsilon = e^{2\pi i/n}$ is as follows:

g_i	1	a^r $(1 \leq r \leq (n-1)/2)$	b
h_i	1	2	n
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leq j \leq (n-1)/2)$	2	$\epsilon^{jr} + \epsilon^{-jr}$	0

Case 2. n even.

If n is even, say $n = 2m$, then the conjugacy classes of D_{2n} are:

$$\{1\}, \ \{a^m\}, \ \{a^r, \ a^{-r}\} \ (1 \leq r \leq m-1), \ \{a^s b \mid s \text{ even}\}, \ \{a^s b \mid s \text{ odd}\}.$$

The character table of D_{2n} (n even, $n = 2m$, $\epsilon = e^{2\pi i/n}$) is given below:

g_i	1	a^m	a^r $(1 \leq r \leq m-1)$	b	ab
h_i	1	1	2	$(n/2)$	$(n/2)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j $(1 \leq j \leq m-1)$	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0

We are now ready to compute the total character for dihedral groups.

Proposition 2.1.

1. The total character τ of D_{2n}, n odd, is:

g_i	1	a^r $(1 \leq r \leq (n-1)/2)$	b
h_i	1	2	n
τ	$n+1$	1	0

2. The total character τ of $G = D_{2n}$, $n = 2m$, is:

g_i	1	a^m	a^r	b	ab
τ	$2(m+1)$	$\begin{cases} 0 & m \text{ odd} \\ 2 & m \text{ even} \end{cases}$	$\begin{cases} 0, & r \text{ odd} \\ 2, & r \text{ even} \end{cases}$	0	0

Proof. When n is odd we clearly have that $\tau(1) = 2 + 2(n-1)/2 = n+1$ and $\tau(b) = 0$. An easy computation yields

$$\tau(a^r) = 2 + \sum_{j=1}^{(n-1)/2} (\epsilon^{jr} + \epsilon^{-jr}) = \frac{1 - \epsilon^{r(n+1)/2}}{1 - \epsilon^r} + \frac{1 - \epsilon^{-r(n+1)/2}}{1 - \epsilon^{-r}}.$$

Recall that $\epsilon = e^{(2\pi i)/n}$ and note that $e^{((n+1)ir\pi)/n} = (-1)^r e^{(ir\pi)/n} = (-1)^r e^{r/2}$ and similarly $e^{-((n+1)ir\pi)/n} = (-1)^r e^{-(ir\pi)/n} = (-1)^r \epsilon^{-r/2}$.

For r odd we get

$$\tau(a^r) = \frac{1 + \epsilon^{r/2}}{1 - \epsilon^r} + \frac{1 + \epsilon^{-r/2}}{1 - \epsilon^{-r}} = \frac{1 + \epsilon^{r/2}}{1 - \epsilon^r} - \epsilon^{r/2} \frac{1 + \epsilon^{r/2}}{1 - \epsilon^r} = 1.$$

Similar calculation for r even yield $\tau(a^r) = 1$ and we have completed the proof of part 1 of Proposition 2.1.

Part 2 of Proposition 2.1 follows easily as above by examining separately the cases for r odd and even.

3. Polynomials of Sharp Characters in Dihedral Groups. Our strategy in proving our main theorem will be to show first that all faithful irreducible characters of dihedral groups of twice odd order are sharp. Then we proceed to construct a polynomial $p(x)$ such that $p(\chi) = \tau$ on all non–identity elements where $\chi = \psi_1$ (using the notation of section 2 for D_{2n} with n odd). We notice first that if such a polynomial exists, then $\tau(b) = 0 = p(\psi(b)) = p(0) = a_0$ and hence, our polynomial will have zero constant term. In addition since $\tau(a^r) = 1$ for all $1 \leq r \leq (n-1)/2$ we must also have that $p(\psi(a^r)) = p(2\cos((2\pi r)/n)) = 1$. The arguments will easily extend to any faithful irreducible character ψ_j of D_{2n} with n odd. The following lemma is mentioned in [1] without proof. We include its proof for completeness.

Lemma 3.1.

$$\prod_{k=1}^{m-1} 2\sin\left(\frac{k\pi}{m}\right) = m.$$

Proof. Let $L = \{e^{(2ki\pi)/m} \mid 1 \leq k \leq m-1\}$. Then,

$$f_L(x) = \prod_{k=1}^{m-1} \left(x - e^{(2ki\pi)/m}\right) = x^{m-1} + \cdots + x + 1.$$

Note that $f_L(1) = m$, which implies that

$$\prod_{k=1}^{m-1} \left(1 - e^{(2ki\pi)/m}\right) = m.$$

Recall that

$$2\sin\left(\frac{k\pi}{m}\right) = \frac{1}{i}\left(e^{(ki\pi)/m} - e^{-(ki\pi)/m}\right) = \frac{i\left(1 - e^{(2ki\pi)/m}\right)}{e^{(ki\pi)/m}}.$$

This implies that

$$\prod_{k=1}^{m-1} 2\sin\left(\frac{k\pi}{m}\right) = \prod_{k=1}^{m-1} \frac{i\left(1 - e^{(2ki\pi)/m}\right)}{e^{(ki\pi)/m}} = \frac{mi^{m-1}}{\prod_{k=1}^{m-1} e^{(ki\pi)/m}}.$$

However,

$$\prod_{k=1}^{m-1} e^{(ki\pi)/m} = e^{(i(m-1)\pi)/2}$$

$$= \left(\cos\left(\frac{m\pi}{2}\right) + i\sin\left(\frac{m\pi}{2}\right)\right)\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right).$$

If $m = 4b$, then $i^{m-1} = i^{4b-1} = i^{4b}i^{-1} = -i$, and

$$\left(\cos\left(\frac{m\pi}{2}\right) + i\sin\left(\frac{m\pi}{2}\right)\right)\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right) = (1+0i)(0-i) = -i.$$

Hence,

$$\prod_{k=1}^{m-1} 2\sin\left(\frac{k\pi}{m}\right) = m.$$

If $m = 4b+1$, then $i^{m-1} = i^{4b} = 1$, and

$$\left(\cos\left(\frac{m\pi}{2}\right) + i\sin\left(\frac{m\pi}{2}\right)\right)\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right) = (0+i)(0-i) = 1.$$

Hence,

$$\prod_{k=1}^{m-1} 2\sin\left(\frac{k\pi}{m}\right) = m.$$

Proceeding similarly for the remaining cases yields the result.

Theorem 3.2. Every faithful irreducible character of D_{2n} (n odd) is sharp.

Proof. Consider ψ_1 as defined previously in the character table of D_{2n}. We show first that ψ_1 is sharp. We must first discern if there is any r, $(1 \le r \le (n-1)/2)$ such that $2\cos((2r\pi)/n) = 2$. Now, $\cos((2r\pi)/n) = 1$ whenever $(2r\pi)/n = 2k\pi$,

$k \in \mathbb{Z}$. Hence, we must have that $r/n = k$, $k \in \mathbb{Z}$, but, $1 \leq r \leq (n-1)/2$. So, $r < n$ implies $r/n \notin \mathbb{Z}$ and we see that $\cos((2r\pi)/n) \neq 1$ for any r such that $1 \leq r \leq (n-1)/2$. Hence, $2\cos((2r\pi)/n) \neq 2$, and ψ_1 is faithful. Now, we have

$$\prod_{r=1}^{(n-1)/2} \left(2 - 2\cos\left(\frac{2r\pi}{n}\right)\right) = \prod_{r=1}^{(n-1)/2} 4\sin^2\left(\frac{r\pi}{n}\right)$$

$$= \prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right) \prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right).$$

Also recall that

$$\sin\left(\pi - \frac{r\pi}{n}\right) = \sin\pi\cos\left(\frac{r\pi}{n}\right)$$

$$-\cos\pi\sin\left(\frac{r\pi}{n}\right) = \sin\left(\frac{r\pi}{n}\right).$$

Thus,

$$\prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right) \prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right) = \prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right) \prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{(n-r)\pi}{n}\right).$$

Reindex the latter product by letting $j = n - r$ and by letting j run from $(n+1)/2$ to $n-1$. So, we now get:

$$\prod_{r=1}^{(n-1)/2} 2\sin\left(\frac{r\pi}{n}\right) \prod_{j=(n+1)/2}^{n-1} 2\sin\left(\frac{j\pi}{n}\right) = \prod_{r=1}^{n-1} 2\sin\left(\frac{r\pi}{n}\right) = n.$$

This is the crucial step in proving that ψ_1 is sharp since

$$f_L(x) = (x-0) \prod_{r=1}^{(n-1)/2} \left(x - 2\cos\left(\frac{2r\pi}{n}\right)\right)$$

and hence, we get $f_L(2) = 2n = |G|$ and ψ_1 must be sharp. Now consider ψ_j with $1 < j \leq (n-1)/2$. It is not difficult to see that ψ_j is faithful if and only if j is relatively prime to n. In that case the set of character values of ψ_j is the same as that of ψ_1 and the same arguments yield that all faithful irreducible characters of D_{2n} with n odd are sharp.

The following lemma is formula JO (570) in [2].

Lemma 3.3. For n odd,

$$n \sin(x) \prod_{k=1}^{(n-1)/2} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)}\right) = \sin(nx).$$

Theorem 3.4.

$$\prod_{k=1}^{(n-1)/2} 2\cos\left(\frac{2k\pi}{n}\right) = \begin{cases} 1, & n \equiv 1 \pmod 8 \\ -1, & n \equiv 3 \pmod 8 \\ -1, & n \equiv 5 \pmod 8 \\ 1, & n \equiv 7 \pmod 8). \end{cases}$$

Proof. Letting $x = \pi/4$ in the formula of Lemma 3.3 above, we get that

$$\sin\left(\frac{n\pi}{4}\right) = n \sin\left(\frac{\pi}{4}\right) \prod_{k=1}^{(n-1)/2} \left(\frac{\sin^2\left(\frac{k\pi}{n}\right) - \frac{1}{2}}{\sin^2\left(\frac{k\pi}{n}\right)}\right),$$

from whence we get

$$\frac{\sin\left(\frac{n\pi}{4}\right)}{n\sin\left(\frac{\pi}{4}\right)} = \frac{\prod_{k=1}^{(n-1)/2}\left(\frac{1}{2}\left(2\sin^2\left(\frac{k\pi}{n}\right) - 1\right)\right.}{\prod_{k=1}^{(n-1)/2}\sin^2\left(\frac{k\pi}{n}\right)}$$

$$= \frac{\prod_{k=1}^{(n-1)/2}(-1)\cos\left(\frac{2k\pi}{n}\right)}{\prod_{k=1}^{(n-1)/2}\left(2\sin^2\left(\frac{k\pi}{n}\right)\right)} = \frac{\prod_{k=1}^{(n-1)/2}(-1)2\cos\left(\frac{2k\pi}{n}\right)}{\prod_{k=1}^{(n-1)/2}4\sin^2\left(\frac{k\pi}{n}\right)}$$

$$= \frac{(-1)^{(n-1)/2}\prod_{k=1}^{(n-1)/2}\left(2\cos\left(\frac{2k\pi}{n}\right)\right)}{\prod_{k=1}^{(n-1)/2}2\sin\left(\frac{k\pi}{n}\right)} = \frac{(-1)^{(n-1)/2}\prod_{k=1}^{(n-1)/2}\left(2\cos\left(\frac{2k\pi}{n}\right)\right)}{n}.$$

This implies that

$$\prod_{k=1}^{(n-1)/2} 2\cos\left(\frac{2k\pi}{n}\right) = (-1)^{(n-1)/2}\frac{\sin\left(\frac{n\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}.$$

If $n = 8k+1$, then $(n-1)/2 = 4k$, and $(-1)^{(n-1)/2} = 1$, and we have that

$$\prod_{k=1}^{(n-1)/2} 2\cos\left(\frac{2k\pi}{n}\right) = \frac{\sin\left(\frac{(8k+1)\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{\sin\left(2k\pi + \frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = 1.$$

Similar arguments for n congruent to 3, 5, 7 $\pmod 8$ complete the proof.

Proposition 3.5. Let $x_r = 2\cos\frac{2r\pi}{n}$ for n odd and $r = 1, \ldots, (n-1)/2$. Let $t = (n-1)/2$. The system

$$\begin{pmatrix} x_1^t & x_1^{t-1} & \cdots & x_1 \\ x_2^t & x_2^{t-1} & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_t^t & x_t^{t-1} & \cdots & x_t \end{pmatrix} \begin{pmatrix} a_t \\ a_{t-1} \\ \vdots \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

has a unique integer solution with

$$a_t = \begin{cases} -1, & n = 1,3 \pmod 8 \\ 1, & n = 5,7 \pmod 8 \end{cases}$$

$$a_{t-r} = (-1)^r a_t S_r(x_1, \ldots, x_t),$$

where $S_r(x_1, \ldots, x_t)$ is the r–th elementary symmetric function in $x_1, x_2, \ldots, x_t$.

Proof. First observe that the uniqueness part follows easily since

$$\det\begin{pmatrix} x_1^t & x_1^{t-1} & \cdots & x_1 \\ x_2^t & x_2^{t-1} & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_t^t & x_t^{t-1} & \cdots & x_t \end{pmatrix} = x_1x_2\cdots x_t \det\begin{pmatrix} x_1^{t-1} & x_1^{t-2} & \cdots & 1 \\ x_2^{t-1} & x_2^{t-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_t^{t-1} & x_t^{t-2} & \cdots & 1 \end{pmatrix}$$

$$= x_1x_2\cdots x_t \prod_{1\le i<j\le t}(x_i - x_j) \neq 0$$

using the well known formula for the Vandermonde determinant. Let $p(x) = a_tx^t + \cdots + a_1x$ such that $p(x_i) = 1$ for all $1 \le i \le t$. Note that $p(x_i) - 1 = 0$ when $1 \le i \le t$. We have $p(x) - 1 = a_t(x - x_1)\cdots(x - x_t)$. By equating the constant terms, we see that $(-1)^t a_t x_1 \cdots x_t = -1$. This in turn yields

$$a_t = \frac{-(-1)^t}{x_1\cdots x_t}.$$

Recall that $t = (n-1)/2$.
If $n \equiv 1 \pmod 8$, then $(-1)^t = 1$ and $x_1 \cdots x_t = 1$. Hence, $a_t = -1$.
If $n \equiv 3 \pmod 8$, then $(-1)^t = -1$ and $x_1 \cdots x_t = -1$. Hence, $a_t = -1$.
If $n \equiv 5 \pmod 8$, then $(-1)^t = 1$ and $x_1 \cdots x_t = -1$. Hence, $a_t = 1$.
If $n \equiv 7 \pmod 8$, then $(-1)^t = -1$ and $x_1 \cdots x_t = 1$. Hence, $a_t = 1$.

The remaining coefficients are clearly given by $a_{t-r} = (-1)^r a_t S_r(x_1, \ldots, x_t)$, where $S_r(x_1, \ldots, x_t)$ is the r–th elementary symmetric function in $x_1, x_2, \ldots, x_t$. Recall that $x_r = \epsilon^r + \epsilon^{-r}$ where ϵ is a primitive n–th root of unity. Therefore all of the x_r and hence, $S_r(x_1, \ldots, x_t)$ are algebraic integers. Note also that $S_r(x_1, \ldots, x_t)$ remains invariant under the Galois automorphisms of $\mathbb{Q}(\epsilon)$ over $\mathbb{Q}$ and is therefore a rational integer. Hence, we have as desired that $a_{t-r} \in \mathbb{Z}$.

As immediate consequences of the above, we get:

Corollary 3.6. Let n be an odd integer, $x_r = 2\cos((2r\pi)/n)$ where $1 \le r \le (n-1)/2$. Let $t = (n-1)/2$.

1. The polynomial

$$p(x) = -(x - x_1)\cdots(x - x_t) + 1, \quad n \equiv 1, 3 \pmod 8$$

has the property that $p(0) = 0$ and $p(x_r) = 1$ for $1 \le r \le t$.

2. The polynomial

$$p(x) = (x - x_1)\cdots(x - x_t) + 1, \quad n \equiv 5, 7 \pmod 8$$

has the property that $p(0) = 0$ and $p(x_r) = 1$ for $1 \le r \le t$.

We are now ready to prove our main Theorem.

Theorem 3.7. Let G be a dihedral group of order $2n$ with n odd. If χ is any faithful irreducible character of G, then a monic polynomial $g(x) \in \mathbb{Z}[x]$ exists such that $g(\chi) = \tau$, where τ is the total character of G. Furthermore $g(x)$ is minimal with this property.

Proof. We need to examine separately the following two cases.

Case 1. $n \equiv 5, 7 \pmod 8$.

Let $\chi = \psi_1$ be the irreducible character of D_{2n} as labeled in its character table previously. Let $x_r = \epsilon^r + \epsilon^{-r} = 2\cos((2r\pi)/n)$, $1 \le r \le (n-1)/2$ and $t = (n-1)/2$. We show that $g(x) = (x - x_1)\cdots(x - x_t) + 1$ is the right polynomial. We have established in Corollary 3.6 part 2, that $g(x) \in \mathbb{Z}[x]$ and that $g(\chi(a^r)) = g(x_r) = 1 = \tau(a^r)$ and $g(\chi(b)) = g(0) = 0 = \tau(b)$. We need only show that $g(\chi(1)) = g(2) = \tau(1) = n + 1$. Notice that since χ is sharp

$$(2-0)\prod_{j=1}^{(n-1)/2}\left(2 - 2\cos\left(\frac{2j\pi}{n}\right)\right) = 2n$$

and hence, we have that

$$\prod_{j=1}^{(n-1)/2}\left(2 - 2\cos\left(\frac{2j\pi}{n}\right)\right) = n.$$

Now it follows easily that

$$g(2) = \prod_{j=1}^{(n-1)/2}\left(2 - 2\cos\left(\frac{2j\pi}{n}\right)\right) + 1 = n + 1 = \tau(1)$$

as desired.

Notice that the result is true for any faithful irreducible character of G since they all share the same set of character values and $\tau(a^r) = 1$ for all r.

Case 2. $n \equiv 1, 3 \pmod 8$.

Let $p(x) = -(x - x_1) \cdots (x - x_t) + 1$ and $g(x) = (p(x) - 1)(-x + 1) + 1$. Notice that by Corollary 3.6 part 1, we have that $p(0) = 0$ and $p(x_r) = 1$ and hence, $g(0) = 0$ and $g(x_r) = 1$. In addition

$$p(2) = -\prod_{j=1}^{(n-1)/2} \left(2 - 2\cos\left(\frac{2j\pi}{n}\right)\right) = -n + 1.$$

Hence, $g(2) = (p(2) - 1)(-2 + 1) + 1 = (-n + 1 - 1)(-1) + 1 = n + 1 = \tau(1)$ as desired. As before the argument can be shown to be independent of the sharp character we choose.

Notice that by construction both polynomials $g(x)$ are minimal with the required properties.

Using GAP and *Mathematica* we can easily compute the polynomials for the first few dihedral groups

$$D_6: \quad x^2$$

$$D_{10}: \quad x^2 + x$$

$$D_{14}: \quad x^3 + x^2 - 2x$$

$$D_{18}: \quad x^5 - 4x^3 + x^2 + 3x$$

$$D_{22}: \quad x^6 - 5x^4 + x^3 + 6x^2 - 2x$$

$$D_{26}: \quad x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x$$

$$D_{30}: \quad x^7 + x^6 - 6x^5 - 5x^4 + 10x^3 + 6x^2 - 4x.$$

4. Final Remarks and Some Questions. It appears that sharpness is instrumental in answering Johnson's question. In fact we are not aware of any instances of a non–sharp character giving an affirmative answer to his question.

In the case of dihedral groups of twice even order we are not always guaranteed an irreducible sharp character. For example, if we look at the character table of D_{12}

g_i	1	a^3	a	a^2	b	ab
h_i	1	1	2	2	3	3
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	1	-1
χ_4	1	-1	-1	1	-1	1
χ_5	2	-2	1	-1	0	0
χ_6	2	2	-1	1	0	0
τ	8	0	0	4	0	0

The only possibility for a sharp character is χ_5. However, $(2-(-2))(2-1)(2-(-1))(2-0) = 4(1)(3)(2) = 24 \neq 12$ and D_{12} has no irreducible sharp characters. It is easy to show that Johnson's question has a negative answer on D_{12}.

It appears however, that if a sharp character χ exists in a dihedral group of twice even order then a polynomial $g(x) \in \mathbb{Z}[x]$ may exist such that $g(\chi) = \tau$ as is the case in D_8 where if χ is the faithful irreducible character of degree 2, then $\chi^2 + \chi = \tau$.

A natural question arising is therefore the following:

Question 4.1. If χ is an irreducible character of a finite group G such that for some monic polynomial $g(x) \in \mathbb{Z}[x]$ we have $g(\chi) = \tau$, is χ necessarily sharp?

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