SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

121. [1998, 176] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Determine the volume of the body obtained by rotating the curve $y = \ln \sin x$, $(0 \le x \le \pi)$ about the x–axis.

Solution by Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin.

$$V = \pi \int_0^{\pi} (\ln \sin x)^2 dx = 2\pi \int_0^{\pi/2} (\ln \sin x)^2 dx.$$

It is known that

$$\int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta = \frac{1}{2}B(a,b) = \frac{1}{2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where B is the beta function and Re $(a) > 0$ and Re $(b) > 0$. A corollary to this result is the following.

<u>Corollary</u>.

$$\int_0^{\pi} \sin^{2\alpha-1} x dx = \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(1/2)}{\Gamma(\alpha+1/2)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)},$$

where α is a positive real number.

Let

$$f(\alpha) = \int_0^{\pi/2} \sin^{2\alpha-1} x dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}.$$

Lemma 1.

$$\frac{d}{d\alpha}\big(\sin^{2\alpha-1}x\big) = 2\sin^{2\alpha-1}x\ln\sin x.$$

Proof. Let $y = \sin^{2\alpha-1}x$. Then $\ln y = \ln(\sin^{2\alpha-1}x) = (2\alpha-1)\ln\sin x$. Thus,

$$\frac{y'}{y} = \frac{d}{d\alpha}\big[(2\alpha-1)\ln\sin x\big] = 2\ln\sin x,$$

so

$$\frac{d}{d\alpha}\big(\sin^{2\alpha-1}x\big) = 2\sin^{2\alpha-1}x\ln\sin x.$$

Lemma 2.

$$\frac{d}{d\alpha}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\cdot\big(\mathrm{psi}(\alpha)-\mathrm{psi}(\alpha+1/2)\big).$$

Proof. Let $y = \Gamma(\alpha)/\Gamma(\alpha+1/2)$. Then, $\ln y = \ln\Gamma(\alpha) - \ln\Gamma(\alpha+1/2)$. Thus,

$$\frac{y'}{y} = \frac{d}{d\alpha}\ln\Gamma(\alpha) - \frac{d}{d\alpha}\ln\Gamma(\alpha+1/2) = \mathrm{psi}(\alpha) - \mathrm{psi}\ (\alpha+1/2),$$

so

$$\frac{d}{d\alpha}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}(\alpha)-\mathrm{psi}(\alpha+1/2)\big).$$

Then,

$$\begin{aligned} f'(\alpha) &= \frac{d}{d\alpha}\int_0^{\pi/2}\sin^{2\alpha-1}x\,dx \\ &= \int_0^{\pi/2}\frac{d}{d\alpha}\big(\sin^{2\alpha-1}x\big)dx = \int_0^{\pi/2}2\sin^{2\alpha-1}x\ln\sin x\,dx \\ &= 2\int_0^{\pi/2}\sin^{2\alpha-1}x\ln\sin x\,dx. \end{aligned}$$

Also,

$$f'(\alpha) = \frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\right)$$

$$= \frac{\sqrt{\pi}}{2}\frac{d}{d\alpha}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\right)$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big).$$

In addition,

$$f''(\alpha) = \frac{d}{d\alpha}\left(2\int_0^{\pi/2} \sin^{2\alpha-1} x \ln\sin x dx\right)$$

$$= 2\int_0^{\pi/2} \frac{d}{d\alpha}\big(\sin^{2\alpha-1} x\big) \ln\sin x dx$$

$$= 2\int_0^{\pi/2} \big(2\sin^{2\alpha-1} x \ln\sin x\big) \ln\sin x dx$$

$$= 4\int_0^{\pi/2} \sin^{2\alpha-1} x \big(\ln\sin x\big)^2 dx.$$

Also,

$$f''(\alpha) = \frac{\sqrt{\pi}}{2}\frac{d}{d\alpha}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big)\right)$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}'(\alpha) - \mathrm{psi}'(\alpha+1/2)\big)\right.$$

$$\left. + \big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big)\frac{d}{d\alpha}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\right)$$

$$= \frac{\sqrt{\pi}}{2}\Bigg(\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}'(\alpha) - \mathrm{psi}'(\alpha+1/2)\big)$$

$$+ \big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big)\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big)\Bigg)$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\Big[\big(\mathrm{psi}(\alpha) - \mathrm{psi}(\alpha+1/2)\big)^2 + \mathrm{psi}'(\alpha) - \mathrm{psi}'(\alpha+1/2)\Big].$$

Hence, if C is Euler's constant,

$$\int_0^{\pi/2} \big(\ln\sin x\big)^2 dx = \frac{1}{4} f''\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\sqrt{\pi}}{2}\frac{\Gamma(1/2)}{\Gamma(1)}\Big[\big(\mathrm{psi}(1/2) - \mathrm{psi}(1)\big)^2 + \mathrm{psi}'(1/2) - \mathrm{psi}'(1)\Big)$$

$$= \frac{1}{4}\frac{\sqrt{\pi}}{2}\frac{\sqrt{\pi}}{1}\left[\big((-C - 2\ln 2) - (-C)\big)^2 + \frac{\pi^2}{2} - \frac{\pi^2}{6}\right]$$

$$= \frac{\pi}{8}\left[4\big(\ln 2\big)^2 + \frac{\pi^2}{3}\right] = \frac{\pi}{2}\left[\big(\ln 2\big)^2 + \frac{\pi^2}{12}\right].$$

Finally,

$$V = \pi\int_0^{\pi} \big(\ln\sin x\big)^2 dx = 2\pi\int_0^{\pi/2} \big(\ln\sin x\big)^2 dx$$

$$= 2\pi\cdot\frac{\pi}{2}\left[\big(\ln 2\big)^2 + \frac{\pi^2}{12}\right] = \pi^2\left[\big(\ln 2\big)^2 + \frac{\pi^2}{12}\right].$$

References

1. M. G. Beumer, "Some Special Integrals," *The American Mathematical Monthly*, 68 (1961), 645–647.
2. F. Bowman, "Note on the Integral $\int_0^{\pi/2}(\log\sin\theta)^n d\theta$," *Journal of the London Mathematical Society,* 22 (1947), 172–173.
3. T. J. l'a Bromwich, *Introduction to the Theory of Infinite Series,* London, Macmillan & Co. Ltd., New York, St. Martin's Press, 1965, 484, 523.
4. Chandhuri, "Some Special Integrals," *The American Mathematical Monthly*, 74 (1967) 545–548.
5. T. V. L. Narasimhan, "A Recursion Formula for a Certain Definite Integral," *The American Mathematical Monthly*, 68 (1961), 993–994.

Also solved by the proposer. One incorrect solution was also received.

122. [1998, 176] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Evaluate

$$\int_0^{+\infty} \frac{\ln^3 x}{\cosh(3\ln x)}dx.$$

Solution I by Joseph Wiener, University of Texas – Pan American, Edinburg, Texas.

Since $\cosh t = (e^t + e^{-t})/2$, the integrand becomes $2x^3\ln^3 x/(1+x^6)$. Let us take an auxiliary integral

$$\int_0^{\infty} \frac{u^{p-1}}{1+u}du = B(p, 1-p) = \frac{\pi}{\sin(p\pi)}, \quad 0 < p < 1,$$

where B denotes Euler's beta function, and differentiate it with respect to the parameter p giving

$$\int_0^\infty \frac{u^{p-1}\ln u}{1+u}du = -\frac{\pi^2\cos(p\pi)}{\sin^2(p\pi)},$$

$$\int_0^\infty \frac{u^{p-1}\ln^2 u}{1+u}du = \pi^3\frac{1+\cos^2(p\pi)}{\sin^3(p\pi)},$$

$$\int_0^\infty \frac{u^{p-1}\ln^3 u}{1+u}du = -\pi^4\frac{\left[5+\cos^2(p\pi\right]\cos(p\pi)}{\sin^4(p\pi)}.$$

In the latter integral, we make the substitution $u = x^6$ which changes the integrand to $6^4 x^{6p-1}\ln^3 x/(1+x^6)$, and it remains to set $6p-1=3$, that is, $p = 2/3$ in order to obtain the value of the original integral: $28(\pi/6)^4/3$.

Solution II by Kenneth B. Davenport, Frackville, Pennsylvania.

To evaluate

$$\int_0^\infty \frac{\ln^3 x}{\cosh(3\ln x)}dx,$$

we let $u = \ln x$ (so $dx = e^u du$). The expression then becomes

$$2\int_{-\infty}^\infty \frac{u^3 e^u du}{e^{3u}+e^{-3u}}. \tag{1}$$

Now note we may evaluate the bounds of the initial expression between 0 to 1 and from 1 to ∞. Thus we may write (1) as

$$2\int_{-\infty}^0 \frac{u^3 e^{4u}}{1+e^{6u}}du + 2\int_0^\infty \frac{u^3 e^{-2u}}{1+e^{-6u}}du. \tag{2}$$

Expanding (2) further we obtain

$$2\left[\int_0^\infty \sum_{k=0}^\infty (-1)^k u^3 e^{-(2+6k)u} du + \int_{-\infty}^0 \sum_{k=1}^\infty (-1)^{k-1} u^3 e^{(6k-2)u} du\right]. \qquad (3)$$

Now using the expansion formula and termwise integration, that is

$$\int x^m e^{ax} dx = \frac{e^{ax}}{a}\left(x^m - \frac{m}{a}x^{m-1} + \frac{m(m-1)}{a^2}x^{m-2}\cdots\right)$$

we have the first expression in (3) becomes

$$\sum_{k=0}^\infty (-1)^k \frac{e^{-(2+6k)u}}{-(2+6k)}\left(u^3 + \frac{3u^2}{2+6k} + \frac{6u}{(2+6k)^2} + \frac{6}{(2+6k)^3}\right)\Bigg|_0^\infty = 12\sum_{k=0}^\infty \frac{(-1)^k}{(2+6k)^4}. \qquad (4)$$

Similarly, the second expression in (3) becomes

$$\sum_{k=1}^\infty (-1)^{k-1} \frac{e^{(6k-2)u}}{6k-2}\left(u^3 - \frac{3u^2}{6k-2} + \frac{6u}{(6k-2)^2} - \frac{6}{(6k-2)^3}\right)\Bigg|_{-\infty}^0 = 12\sum_{k=1}^\infty -\frac{(-1)^{k-1}}{(6k-2)^4}. \qquad (5)$$

Now we use the fact (from Abramowitz and Stegun, p. 808) that

$$\eta(4) = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \cdots = \frac{7\pi^4}{720}.$$

The sum of the series, (4) and (5), is then

$$12\left(\frac{\eta(4)}{16} - \frac{\eta(4)}{1296}\right) = \frac{7\pi^4}{972}.$$

Solution III by Bob Prielipp, University of Wisconsin – Oshkosh, Oshkosh, Wisconsin.

It is known [1] that for $0 < j + 1 < n$,

$$\int_0^\infty \frac{x^j dx}{x^n + a} = \frac{\pi}{n a^{(n-j-1)/n} \sin[(j+1)\pi/n]}.$$

Thus,

$$\int_0^\infty \frac{x^{m-1}}{x^n + 1} dx = \frac{\pi}{n \sin(m\pi/n)}, \quad 0 < m < n.$$

Letting $x = u^6$ (so $dx = 6u^5 du$), $6m - 1 = t$, and $n = 1$ yields

$$I(t) = \int_0^\infty \frac{u^t}{u^6 + 1} du = \frac{\pi}{6 \sin((t+1)\pi/6)}, \quad -1 < t < 5.$$

Lemma 1.

$$2I'''(3) = 2 \int_0^\infty \frac{(\ln u)^3}{u^6 + 1} u^3 du.$$

Proof. Let $y(t) = u^t$. Then $\ln y = t \ln u$ so $y'/y = \ln u$. Thus, $y'(t) = u^t \ln u$. Hence,

$$I'(t) = \int_0^\infty \frac{d}{dt} \left(\frac{u^t}{u^6 + 1} \right) du = \int_0^\infty \frac{1}{u^6 + 1} \frac{d}{dt} (u^t) du$$

$$= \int_0^\infty \frac{1}{u^6 + 1} u^t \ln u du = \int_0^\infty \frac{\ln u}{u^6 + 1} u^t dt.$$

It follows that

$$I''(t) = \int_0^\infty \frac{\ln u}{u^6+1}\frac{d}{dt}(u^t)dt = \int_0^\infty \frac{\ln u}{u^6+1}u^t \ln u du = \int_0^\infty \frac{(\ln u)^2}{u^6+1}u^t du$$

and that

$$I'''(t) = \int_0^\infty \frac{(\ln u)^2}{u^6+1}\frac{d}{dt}(u^t)du = \int_0^\infty \frac{(\ln u)^2}{u^6+1}u^t \ln u du = \int_0^\infty \frac{(\ln u)^3}{u^6+1}u^t du.$$

Therefore,

$$2I'''(3) = 2\int_0^\infty \frac{(\ln u)^3}{u^6+1}u^3 du.$$

Lemma 2.

$$2\int_0^\infty \frac{u^3(\ln u)^3}{u^6+1}du = \int_0^\infty \frac{(\ln x)^3}{\cosh(3\ln x)}dx.$$

Proof. Because $2\cosh w = e^w + e^{-w}$, it follows that

$$\int_0^\infty \frac{(\ln x)^3}{2\cosh(3\ln x)}dx = \int_0^\infty \frac{(\ln x)^3}{e^{3\ln x}+e^{-3\ln x}}dx = \int_0^\infty \frac{(\ln x)^3}{x^3+x^{-3}}dx$$

$$= \int_0^\infty \frac{x^3(\ln x)^3}{x^6+1}dx = \int_0^\infty \frac{u^3(\ln u)^3}{u^6+1}du.$$

Hence,

$$2\int_0^\infty \frac{u^3(\ln u)^3}{u^6+1}du = \int_0^\infty \frac{(\ln x)^3}{\cosh(3\ln x)}dx.$$

Let

$$z(t) = \frac{\pi}{6}\left(\sin\frac{(t+1)\pi}{6}\right)^{-1}.$$

Then,

$$\begin{aligned}
z'(t) &= \frac{\pi}{6}(-1)\left(\sin\frac{(t+1)}{6}\right)^{-2}\cos\frac{(t+1)\pi}{6}\cdot\frac{\pi}{6} \\
&= -\left(\frac{\pi}{6}\right)^{2}\left(\sin\frac{(t+1)\pi}{6}\right)^{-2}\cos\frac{(t+1)\pi}{6},
\end{aligned}$$

$$\begin{aligned}
z''(t) = -\left(\frac{\pi}{6}\right)^{2}\Bigg[&\left(\sin\frac{(t+1)\pi}{6}\right)^{-2}\left(-\sin\frac{(t+1)\pi}{6}\right)\frac{\pi}{6} \\
&+\cos\frac{(t+1)\pi}{6}(-2)\left(\sin\frac{(t+1)\pi}{6}\right)^{-3}\cos\frac{(t+1)\pi}{6}\cdot\frac{\pi}{6}\Bigg] \\
=\left(\frac{\pi}{6}\right)^{3}\Bigg[&\left(\sin\frac{(t+1)\pi}{6}\right)^{-1}+2\left(\sin\frac{(t+1)\pi}{6}\right)^{-3}\left(\cos\frac{(t+1)\pi}{6}\right)^{2}\Bigg],
\end{aligned}$$

and

$$
\begin{aligned}
z'''(t) = \left(\frac{\pi}{6}\right)^3 \Bigg[(-1)\left(\sin\frac{(t+1)\pi}{6}\right)^{-2} \cos\frac{(t+1)\pi}{6} \cdot \frac{\pi}{6} \\
+ 2\left(\sin\frac{(t+1)\pi}{6}\right)^{-3} \cdot 2\left(\cos\frac{(t+1)\pi}{6}\right)\left(-\sin\frac{(t+1)\pi}{6}\right)\frac{\pi}{6} \\
+ \left(\cos\frac{(t+1)\pi}{6}\right)^2 \cdot 2(-3)\left(\sin\frac{(t+1)\pi}{6}\right)^{-4} \cos\frac{(t+1)\pi}{6} \cdot \frac{\pi}{6} \Bigg] \\
= \left(\frac{\pi}{6}\right)^4 \Bigg[-\left(\sin\frac{(t+1)\pi}{6}\right)^{-2} \cos\frac{(t+1)\pi}{6} \\
- 4\left(\sin\frac{(t+1)\pi}{6}\right)^{-2} \cos\frac{(t+1)\pi}{6} \\
- 6\left(\sin\frac{(t+1)\pi}{6}\right)^{-4} \left(\cos\frac{(t+1)\pi}{6}\right)^3 \Bigg] \\
= \left(\frac{\pi}{6}\right)^4 \Bigg[-5\left(\sin\frac{(t+1)\pi}{6}\right)^{-2} \cos\frac{(t+1)\pi}{6} \\
- 6\left(\sin\frac{(t+1)\pi}{6}\right)^{-4} \left(\cos\frac{(t+1)\pi}{6}\right)^3 \Bigg].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
z'''(3) &= \left(\frac{\pi}{6}\right)^4 \left[-5\left(\frac{\sqrt{3}}{2}\right)^{-2} \left(-\frac{1}{2}\right) - 6\left(\frac{\sqrt{3}}{2}\right)^{-4} \left(-\frac{1}{2}\right)^3 \right] \\
&= \left(\frac{\pi}{6}\right)^4 \left[\frac{5}{2} \cdot \frac{4}{3} + \frac{3}{4} \cdot \frac{16}{9} \right] = \frac{\pi^4}{1296}\left(\frac{10}{3} + \frac{4}{3}\right) \\
&= \frac{\pi^4}{1296} \cdot \frac{14}{3} = \frac{7\pi^4}{648 \cdot 3},
\end{aligned}
$$

making

$$2z'''(3) = \frac{7\pi^4}{324 \cdot 3} = \frac{7\pi^4}{972}.$$

Therefore, the value of the given integral is

$$\frac{7\pi^4}{972}.$$

<u>*References*</u>

1. O. J. Farrell and B. Ross, "Note on Evaluating Certain Real Integrals by Cauchy's Residue Theorem," *The American Mathematical Monthly*, (1961), 151–152.

Also solved by the proposer.

123. [1998, 176] *Proposed by Mohammad K. Azarian, University of Evansille, Evansville, IN.*

If $x_{n+2}/x_{n+1} = x_n$, $n \geq 0$, $x_0 = 1$, and $x_1 = e$ (e is the base for the natural logarithm), then find

$$\lim_{n\to\infty} \frac{\ln\left(\prod_{i=0}^{n} x_{2i+1}\right)}{\ln x_{2n+1}}.$$

Solution by Jim Vandergriff, Austin Peay State University, Clarksville, Tennessee; Joseph Wiener, University of Texas – Pan American, Edinburg, Texas; Reiner Martin, Deutsche Bank, Sevenoaks, Kent, England; N. J. Kuenzi, University of Wisconsin – Oshkosh, Oshkosh, Wisconsin; Carl Libis, Antioch College, Yellow Springs, Ohio; Kandasamy Muthuvel, University of Wisconsin – Oshkosh, Oshkosh, Wisconsin; Jayanthi Ganapathy, University of Wisconsin – Oshkosh, Oshkosh, Wisconsin; and the proposer.

$$\lim_{n\to\infty} \frac{\ln\left(\prod_{i=0}^{n} x_{2i+1}\right)}{\ln x_{2n+1}} = \lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \frac{1+\sqrt{5}}{2},$$

where F_n is the nth Fibonacci number.

Proof. Note that

$$x_0 = e^0,\ x_1 = e^1,\ x_2 = e^1,\ x_3 = e^2,\ x_4 = e^3,\ x_5 = e^5,\ \ldots,\ x_n = e^{F_n},$$

where F_n is the nth Fibonacci number. ($F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.) Then $\ln x_n = F_n$. Thus, for $n \geq 0$, we have

$$\frac{\ln\left(\prod_{i=0}^{n} x_{2i+1}\right)}{\ln x_{2n+1}} = \frac{\sum_{i=0}^{n}\left(\ln x_{2i+1}\right)}{\ln x_{2n+1}} = \frac{\sum_{i=0}^{n} F_{2i+1}}{F_{2n+1}}.$$

Using the well–known identity,

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2},$$

we have

$$\frac{\sum_{i=0}^{n} F_{2i+1}}{F_{2n+1}} = \frac{F_{2n+2}}{F_{2n+1}}.$$

Using another well–known identity,

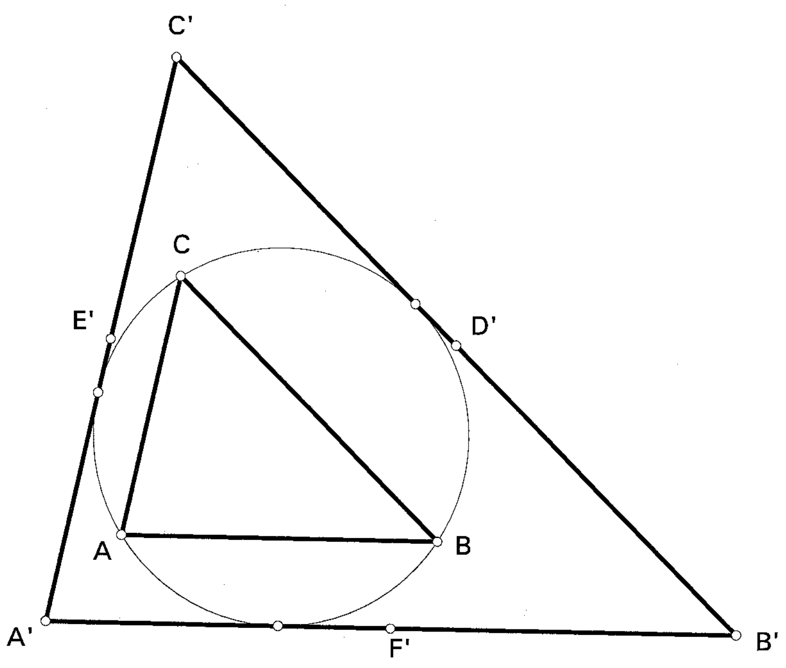
$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2},$$

we have

$$\lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

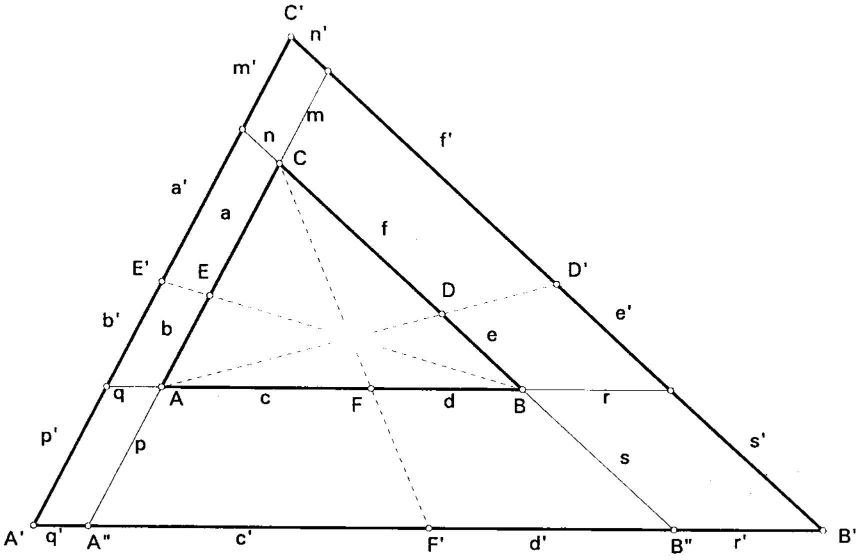
124. [1998, 176; 1999, 45–46] *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, Tennessee.*

Let $\triangle ABC$ be inscribed in a circle and $\triangle A'B'C'$ be circumscribed about the same circle such that the corresponding sides are parallel. Let D', E', and F' be the midpoints of sides $B'C'$, $C'A'$, and $A'B'$, respectively. Prove that AD', BE', and CF' are concurrent. (See figure on next page.)



Solution I by the proposer.

The circle in the problem is irrelevant to its solution. It was included to show how the problem originated. In the figure on the next page, we omit the circle for clarity and label the segments as shown.



Our goal is to use Ceva's theorem; that is, if we can show that

$$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = 1$$

in $\triangle ABC$, then AD, BE, and CF are concurrent.

(1) Since lines AB and $A'B'$ are parallel for $\triangle ABC$ and since D' and E' are midpoints, then

$$\frac{s}{p} = \frac{s'}{p'} = \frac{n' + f' + e'}{m' + a' + b'} = \frac{(e' + s') + e'}{(b' + p') + b'} = \frac{2e' + s'}{2b' + p'}.$$

Hence, $p(2e' + s') = s(2b' + p')$ which can be rewritten as

$$\frac{s}{p} = \frac{e'}{b'},$$

since $ps' = p's$. In the same manner

$$\frac{m}{r} = \frac{a'}{d'} \text{ and } \frac{q}{n} = \frac{c'}{f'}.$$

(2) Since $AB \parallel A'B'$ for $\triangle ABC$, then

$$\frac{a+b}{p} = \frac{f+e}{s}$$

which can be rewritten as

$$\frac{s}{p} = \frac{e+f}{a+b}.$$

Similarly,

$$\frac{q}{n} = \frac{c+d}{e+f} \text{ and } \frac{m}{r} = \frac{a+b}{c+d}.$$

(3) Since $\triangle AFC \sim \triangle A''F'C$, $\triangle FBC \sim \triangle F'B''C$, and lines AB and $A'B'$ are parallel for $\triangle ABC$, then

$$\frac{c}{c'} = \frac{a+b}{p}, \quad \frac{f+e}{s} = \frac{d}{d'}, \text{ and } \frac{a+b}{p} = \frac{f+e}{s}.$$

Hence

$$\frac{c}{c'} = \frac{d}{d'}, \text{ or } \frac{c}{d} = \frac{c'}{d'}.$$

Similarly,

$$\frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{e}{f} = \frac{e'}{f'}.$$

(4) Therefore, using steps 3, 1, and 2 and some algebra, we have

$$\begin{aligned}
\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f} &= \frac{a'}{b'}\cdot\frac{c'}{d'}\cdot\frac{e'}{f'} \\
&= \frac{a'}{d'}\cdot\frac{c'}{f'}\cdot\frac{e'}{b'} \\
&= \frac{m}{r}\cdot\frac{q}{n}\cdot\frac{s}{p} \\
&= \frac{a+b}{c+d}\cdot\frac{c+d}{e+f}\cdot\frac{e+f}{a+b} \\
&= 1.
\end{aligned}$$

Therefore, by Ceva's theorem, AD', BE', and CF' are concurrent.

Solution II by the proposer.

Since D', E', and F' are midpoints of the sides of $\triangle A'B'C'$, then $\triangle D'E'F'$ has its sides parallel to corresponding sides of $\triangle A'B'C'$, and hence also parallel to corresponding sides of $\triangle ABC$. Therefore, lines AB and $D'E'$, lines BC and $E'F'$, and lines CA and $D'F'$ meet in three ideal points which are collinear on an ideal line.

By Desargues' theorem, which says that if two triangles are situated so that lines joining corresponding vertices of the triangles are concurrent, then the corresponding sides of the two triangles meet in three points which are collinear, and conversely, we have that the lines AD', BE', and CF' are concurrent.

Solution III by Clayton W. Dodge, University of Maine, Orono, Maine.

It is known that triangle $D'E'F'$ has its sides parallel to and exactly half the length of the corresponding sides of triangle $A'B'C'$. Thus they are parallel to those of triangle ABC and hence triangles ABC and $D'E'F'$ are directly similar. Furthermore, unless triangle ABC is equilateral, triangle $D'E'F'$ is larger than triangle ABC since its vertices lie on or outside the circumcircle of triangle $A'B'C'$ with at least one vertex outside. It is known and easy to prove that if two triangles are similar and have corresponding sides parallel and either they are not congruent or a $180°$ rotation carries one to the other, then the three lines joining their corresponding vertices concur. When the two triangles are congruent and one is a translation of the other, then the three joins of their corresponding vertices are parallel and not concurrent. This situation cannot occur here, since even when triangle ABC is equilateral, then it is a $180°$ rotation and not a translation that maps triangle ABC to triangle $D'E'F'$. Hence the stated lines concur.

To prove the theorem stated above, let BE' and CF' meet at K. Then triangles KBC and $KE'F'$ are similar and their ratio of similarity is $r = BC/E'F'$. Now on line KA locate point D'' so that K is between A and D'' and $KA/KD'' = r$. By the similar triangles formed it follows that $D'' \equiv D'$. That is, AD', BE', and CF' meet at K.