MEFTAH AL–HESAB: A SUMMARY

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Introduction. *Meftah al–hesab* ("The Key of Arithmetic" or "The Calculator's Key") was written in Arabic by Jamshid Kashani. In the preface of *Meftah al–hesab*, Kashani (Al–Kashani) introduced himself as Jamshid–ibn–e Masud–ibn–e Mahmud–e Tabib–e Kashani mulaqqab be Ghiyath ud-din; that is, "Jamshid son of Masud, Masud son of Mahmud, a physician from the city of Kashan, known as Ghiyath ud–din." Although Kashani was a physician, his main interest was in mathematics and astronomy. Kashani's exact birth date is not known, but it was sometime during the second half of the fourteenth century in the city of Kashan in central Iran. Kashani died on the morning of Wednesday June 22, 1429 (Ramadan 19, 832 A. H. L.) outside Samarqand at the observatory he had helped to build in Uzbekistan. (For the start of the Islamic calendar and the lunar months, see the remark just before the acknowledgment.)

Meftah al–hesab is Kashani's best known work. It contains some of Kashani's original findings, as well as his improvements on earlier Muslim mathematicians' works. Written primarily as a textbook, *Meftah al–hesab* was used for more than five centuries, not only as a textbook, but also as an encyclopedia; it served many generations of students, accountants, astronomers, architects, engineers, land surveyors, merchants, and other professionals. It took Kashani more than seven years to complete *Meftah al–hesab*, which he did on March 2, 1427 (Jamadi al–awwal 3, 830 A. H. L.). To support himself financially, he dedicated this unique mathematical masterpiece to Sultan Ulugh Beg, grandson of Timur and the ruler of Samarqand.

An English summary of *Meftah al–hesab*, like this present paper, is long overdue. Original handwritten copies of *Meftah al–hesab* can be found in libraries throughout the world. The most authentic manuscript of *Meftah al–hesab* was written by Abdul Razzaq–ibn–e Mohammad, known as Muein ud–din Kashi, just two months after its completion. This copy is preserved at the Ketabkhaneh–e Melli–e Malek ("National Library of Malek") in Iran as part of the collection number 3180. Also in this library, another handwritten copy of *Meftah al–hesab* exists under collection Number 3252. Some other handwritten copies of *Meftah al–hesab* which exist in Iran are: Numbers 445 and 165 at the Central Library of the Astan–e Quds–e Razavi (in Mashhad); Numbers 2066, 1790.1, and 6287 at the Central Library of the University of Tehran; Number 442.1 at the School of Literature of the University of Tehran; Number 1530 (1519) of the Library at the Majlis–e showra–e Melli.

Meftah al-hesab has been translated in part or in whole with comments into other languages. A. Dakhel [4] used the Princeton University copy as well as other copies to write an English translation with commentary of the fifth chapter of the third book. N. Rajaei [8], also has written her doctoral thesis in English on the invention of decimal fractions based on *Meftah al-hesab* and *Resaleh–e Muhitiyyeh.* Thus, other than the fifth chapter of the third book, the rest of *Meftah al–hesab*, namely the other thirty–six chapters in its five books, are neither translated nor have comments in English. F. Woepcke [12] translated part of the preface as well as some selected rules (formulas) of *Meftah al–hesab* into French. P. Luckey in his book [5] translated the preface as well as the introduction into German, and discussed some selected parts of *Meftah al–hesab.* Also, P. Luckey's article [6] concerning the extraction of the nth root of an integer and the binomial expansion in Islamic mathematics is based on *Meftah al–hesab.* A. Qurbani in his book [7] explained and commented on most of *Meftah al–hesab* topics in Persian, and translated the entire preface from Luckey's German translation into Persian. Qurbani used Arabic manuscripts as well as English, French, German, Persian, and Russian sources. Qurbani's book has been the main source for this article. Most examples and equations presented here can be found in Qurbani's book as well. M. Tabatabaei translated a few lines of the preface into Persian also. B. A. Rosenfeld's translation of *Meftah al–hesab* into Russian, along with commentaries by A. P. Youschkevitch, can be found in [9]. In his 1956 book, B. A. Rosenfeld included the Arabic text with his revised Russian translation with extended commentaries, and included an exposition by M. Chelebi (grandson of both Qazi Zadeh–e Rumi and Ali Qushji).

Meftah al–hesab consists of a preface, an introduction, and five books. In the introduction Kashani defines the terms *arithmetic, fraction*, and various types of numbers (throughout this paper number means "non–negative integer", unless otherwise noted). In the remainder of this paper we are going to discuss some content of each of the five books. The reader is reminded that this article is by no means a translation of *Meftah al–hesab.* In this paper we present those topics and ideas from each book of *Meftah al–hesab* which we believe are mathematically significant, and easy to communicate in modern notations without getting into lengthy discussions and complicated calculations.

Book I: On the Arithmetic of Integers. Book I of *Meftah al–hesab* consists of six chapters. In chapter one, Kashani did not recognize zero as a number or a digit. In chapter two, he still did not recognize zero as a digit, and said that zero does not have a half. Nonetheless, in dividing zero by two, he wrote zero as the quotient. Finally, in chapter three he said that the product of zero by any other number is zero, and thus he treated zero as a number. Also, in chapter three he defined multiplication of two numbers as, "To multiply two numbers is to obtain a number such that its ratio to one of the numbers is equal to the ratio of the other number to the unit." He then gave a rule for such multiplication, and supported his argument by examples. He used networks to obtain multiplication results. His method of multiplication differed from the modern way of multiplying two numbers.

However, he was aware of the modern method, and he used it later in dealing with the quotient of two numbers. He thought his method was easier to comprehend for beginners than the methods used nowadays. In the preface of *Meftah al–hesab*, Kashani acknowledged that the above method of multiplication was not his own. As an example, he obtained the product of 7806 by 175 using this method (Table 1). In this table, the first column (right hand side) is the product of 6 by 1, 7, and 5, respectively. The second column is the product of 0 by 1, 7, and 5. Columns three and four are defined analogously. To find the result of the product, he added the numbers in each slant strip like nowadays' addition, carrying out the 10 as 1 to the second slant strip, carrying 20 as 2 to the next slant strip, and so on. This method, with a slight change, can be found in *Talkheith–e aamal al–hesab* written by the Arab–Spanish mathematician ibn–e Al–bana–e Marakeshi about a century before Kashani's time. Kashani also presented his own multiplication network with a slight variation and called it *shabakeh–e muvarrab* "slant network." In addition, he presented his method without using a network. He illustrated his network method by finding the product of 624 by 358 (Table 2). Other than its appearance, the idea and the method used in Table 2 are the same as that of Table 1.

In the beginning of chapter four he defined the quotient of two numbers as, "To divide two numbers is to obtain a number such that its ratio to the unit is equal to the ratio of the dividend to the divisor, or to obtain a number such that its ratio to the dividend is the same as the ratio of the unit to the divisor." He then presented a general rule for division which he obtained from his Muslim mathematician forefathers, yet which he himself improved. For purposes of illustration, he divided 3565908 by 475 in two slightly different ways. Finally, he discussed his own methods (two methods) of division, which were very similar to the modern method, and gave examples.

Chapter five is devoted to the extraction of the nth root. In this chapter, Kashani defined the exponent and the root of a number, and defined many other terms related to the exponent of a number. He improved his predecessors' method of finding square roots and presented two methods which were similar to the modern way of extracting square roots. To find the square root of a perfect square he used the identity

$$(a+b+c+\cdots)^2 = a^2 + (2a+b)b + (2a+2b+c)c + \cdots . \tag{1}$$

To approximate the square root of a whole number X which is not a perfect square, his method led to the formula

$$\sqrt{X} = \sqrt{T^2+r} \cong T + \frac{r}{2T+1}. \tag{2}$$

Many Iranian mathematicians as well as other Muslim mathematicians worked on the extraction of the nth root prior to Kashani, including: Aub–Alwafa–e Buzjani and Kushaiar ibn–e Gilani in the fourth century A. H. L.; Abu–Rayhan–e Beiruni, Mohammad Ibn–e Ayyub–e Tabari, Ali ibn–e Ahmad–e Nasvi, and Omar Khayyam–e Nishabouri in the fifth century A. H. L.; Nasir ud–din–e Tusi in the seventh century A. H. L.; and Nezam ud–din–e Aaraj in the eighth century A. H. L. It was Kashani, however, who described in detail a general method for extracting the nth root of a whole number in the decimal system as well as the sexagesimal system (base 60). More than four centuries later, in the nineteenth century, Kashani's method was rediscovered by European mathematicians, and now is known as Ruffini–Horner's method. According to A. Juschkewitsch [13], "The only known general rule for the extraction of the nth root of a whole number ever written by a Muslim mathematician is the one in *Meftah al–hesab*." However, based on Karine Chemla's paper [3], Kashani was not the originator of the Ruffini–Horner's method for extracting roots. To approximate the nth root Kashani used the formula

$$\sqrt[n]{X} = \sqrt[n]{T^n + r} \cong T + \frac{r}{(T+1)^n - T^n}, \tag{3}$$

which is a generalization of his square root approximation (formula (2)). At the conclusion of chapter five of Book I, Kashani stated that he had another method for the extraction of the nth root, and that he would publish it in a separate treatise. Unfortunately, no record of such a treatise exists.

Toward the end of chapter five, he gave a general rule identical to that of Newton for the expansion of $(a + b)^n$, and he made use of a triangular table to find the coefficients in this expansion. Nowadays, this triangle is called the Pascal triangle. Qurbani in his book [7] claimed that the binomial expansion as well as the Pascal triangle both were discovered by the Persian mathematician, astronomer, scientist, and poet Omar Khayyam Nishabouri about six centuries before the era of Newton and Pascal. But, we know that both the binomial expansion and the Pascal triangle were known before him by Karaji (Al–Karaji) in Persia and by various mathematicians in China (and even possibly in India). At the end of chapter five, Kashani presented a method for calculation of $a^n - b^n$, and he extended this method to find an expansion for $(a + b)^n - b^n$. He then applied this method to calculate $(T + 1)^n - T^n$ in the extraction of the nth root formula. Finally, in chapter six, he described the "casting out 9" method for checking the accuracy of multiplication as well as division and root extraction. It is apparent from *Meftah al–hesab* that Kashani had a better grasp of this procedure than his forefathers.

Book II: On the Arithmetic of Fractions. Book II consists of twelve chapters. In the beginning of the first chapter Kashani defined several types of fractions in the decimal system and hinted at sexagesimal fractions. In some places he used the word "fraction" for both the numerator of the fraction as well as the fraction itself. He then categorized fractions, gave a name to each category, and presented rules for the arithmetic of fractions. For instance, he called all fractions, with one as the numerator *kasr–e mujarrad* ("single fraction"). Kashani's inspiration for sexagesimal fractions led him to the discovery of decimal fractions in 1423 (for further detail see [1]). In the second chapter, he discussed ways of writing a fraction, and introduced symbols for addition, subtraction, multiplication, and division of fractions. He devoted the third chapter to the discussion of the greatest common divisor (GCD) of two natural numbers, and he presented a method for finding the GCD of two numbers by repeated use of division. Basically, his method was equivalent to the Euclidean Algorithm (Proposition 2 of book seven of the *Elements*). In chapter four, he dealt with mixed and improper fractions as well as their conversion from mixed to improper and vice versa. In the fifth chapter, he presented two methods for finding the least common denominator (LCD) for a sum of fractions. These two methods differed slightly, but both were very similar to our current methods of finding the LCD. The sixth, seventh, eighth, and ninth chapters all were devoted to the arithmetic of decimal fractions.

In chapter ten, for two whole numbers a and b he used the formula

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{b^{n-1}a}}{b}, \tag{4}$$

for the extraction of the nth root of the fraction $a/b < 1$. Then, he used formula (3) to approximate the numerator $b^{n-1}a$ in equation (4). He made use of both (3) and (4) to approximate the nth root of the fraction $a/b > 1$. In order to get a better approximation, he advised to change mixed fractions into improper fractions, prior to approximation. In chapter eleven he presented a method for rewriting a given fraction with a desired denominator. Finally, in chapter twelve he introduced consumer fractions such as 1/6, 1/24, 1/96, etc., and gave rules for multiplication and division of these fractions.

Book III: On the Computation of Astronomers. All six chapters of this book are concerned with the sexagesimal system, which was useful for computation in astronomy. Kashani discussed *hesab–e jamal* ("jamal arithmetic") in the first chapter, and like other medieval Muslim mathematicians he used it in his mathematical arguments. In *jamal arithmetic* the 28 letters of the Arabic alphabets are used in a specified order (different than alphabetical order) to represent 1's, 10's and 100's. Kashani used a combination of *jamal* letters to represent a number in the sexagesimal system.

Chapter one dealt with the writing of numbers in sexagesimal system. The earliest Persian mathematician to write a book on sexagesimal fractions was Kushaiar ibn–e Gilani. Muslim mathematicians before Kashani used to use a combination of decimal and sexagesimal systems in products as well as quotients of numbers. The Greeks did not use a pure sexagesimal system either. For instance, Ptolemy wrote the length of the year in terms of days as a combination of decimal and sexagesimal system as:

$$365 \quad 14 \quad 48, \qquad \text{which abbreviates} \qquad 365 + \frac{14}{60} + \frac{48}{(60)^2}.$$

However, all discussions and all arithmetic in the third book of *Meftah al–hesab* are done in pure sexagesimal system, including the extraction of the nth root. Thus, Kashani was the first Muslim mathematician to work in a purely sexagesimal system or purely decimal system, without combining the two. From his discussion in *Meftah al–hesab* it is apparent that the rules of arithmetic in sexagesimal system reached their final form during Kashani's time. It is noteworthy that Kashani was the first Muslim mathematician to use zero as an exponent, as in $60^0 = 1$. He also used negative exponents in dealing with minutes (60^{-1}), seconds (60^{-2}), and other numbers smaller than a second.

The fifth chapter is devoted to the extraction of roots. He presented a general rule for the extraction of the nth root in the sexagesimal system, and illustrated his method by finding the following roots (to write a number in base 60 with modern notation, we separate digits by commas, and the integral and the fractional parts by a semicolon):

$$\sqrt{10,9,49,20} = 24,41;40,$$

$$\sqrt{18,52;59,43,51,25} = 10;25,30,$$

$$\sqrt[6]{34,59,1,7,14,54,23,3,47,37;40} = 14,0;30.$$

In the last chapter of the third book, Kashani continued his discussion of decimal fractions before giving rules for conversion of sexagesimal fractions into decimal fractions.

Book IV: On the Measurement of Plane Figures and Bodies. This book contains nine chapters and an introduction. It is concerned with the measurement of dimensions, areas, and volumes of geometric solids. Once again Kashani showed his calculation skills in finding the areas of regular n–gons as well as the volumes of regular polyhedra in both the sexagesimal and the decimal systems. At the conclusion of the seventh chapter, Kashani indicated that he obtained the rules for the calculations of volumes of solids from Khajeh Nasir ud–din Tusi's book called *Ousul–e Ouqlidusi* ("Euclidean Foundations"). Kashani created several tables in the sexagesimal system as well as the decimal system, including a table for the volume of solids, and a table for the values of *jaibs* (the *jaib* of an angle α is the number $60 \sin \alpha$, where α ranges from $0°$ to $90°$ in one–degree steps) and gave instructions for their usage. He defined the center of a triangle and gave the definitions of a rhombus, parallelogram, right trapezoid, quadrilateral with sides of different lengths, and two other quadrilaterals. Essentially all definitions were the same as the modern definitions. He also gave definitions regarding the perpendicular bisector of a chord of a circle, the area between two concentric circles, and other shapes involving two intersecting circles.

Chapter three concerns regular polygons. Kashani referred to the diameter of the circumscribed circle as the *qutr–e atval* ("the long diameter") of the regular n–gon, and the diameter of the inscribed circle as the *qutr–e aqsar* ("the short diameter") of the regular n–gon. To compute the radius R_n of the circumscribed circle, the radius r_n of the inscribed circle, and the area S_n of a regular n–gon with sides of length x_n he presented the following formulas:

$$R_n = \frac{x_n}{2 \sin\left(\frac{180}{n}\right)^\circ},$$

$$r_n = \frac{x_n}{2} \cot\left(\frac{180}{n}\right)^\circ,$$

and

$$\frac{S_n}{x_n^2} = \frac{n}{4} \cot\left(\frac{180}{n}\right)^\circ.$$

To calculate S_3, the area of an equilateral triangle with sides of length x_3, and areas of regular n–gons ($n = 5$–$10, 12, 15, 16$) he used the formula $(n/4)\cot(180/n)^\circ$ to produce two separate tables in sexagesimal and decimal systems. If we represent the area of a regular n–gon by S_n ($n = 5$–$10, 12, 15, 16$), then his table in modern base 10 notation can be summarized as:

$$S_3 = 0.433012x_3^2,$$

$$S_5 = 1.720477x_5^2,$$

$$S_6 = 2.598076x_6^2,$$

$$S_7 = 3.633914x_7^2,$$

$$S_8 = 4.828447x_8^2,$$

$$S_9 = 6.181825x_9^2,$$

$$S_{10} = 7.694909x_{10}^2,$$

$$S_{12} = 11.196152x_{12}^2,$$

$$S_{15} = 17.642363x_{15}^2,$$

$$S_{16} = 20.109358x_{16}^2.$$

For an equilateral triangle with an altitude h Kashani obtained the following formula for the area as well:

$$S_3 = \sqrt{3\left(\frac{x_3}{2}\right)^4} = \sqrt{\frac{h^4}{3}},$$

where $h = \sqrt{(3x_3^2)/4}$. Regarding hexagons Kashani gave the following formulas:

$$S_6 = \sqrt{12r_6^4},$$

$$S_6 = \frac{\sqrt{27x_6^4}}{2},$$

$$2r_6 = \sqrt{3x_6^2},$$

$$(S_6)^2 = x_6^3(6x_6)(1 + 1/8).$$

Similarly he obtained the following formulas for octagons:

$$S_8 = (2r_8)^2 - x_8^2 = 2x_8^2 + 2x_8\sqrt{2x_8^2} = 2x_8^2(1 + \sqrt{2}),$$

$$2r_8 = \sqrt{2x_8^2} + x_8 = x_8(\sqrt{2} + 1),$$

$$x_8 = \sqrt{2(2r_8)^2} - 2r_8 = 2r_8(\sqrt{2} - 1).$$

For a scalene triangle ABC with sides of lengths a, b, and c, and the inscribed circle of radius r with $p = (a + b + c)/2$ he obtained

$$S_3 = \sqrt{p(p-a)(p-b)(p-c)},$$

$$S_3 = \frac{r_3(a + b + c)}{2},$$

$$AH = h_a = c \sin B,$$

$$BH = \frac{a^2 - b^2 + c^2}{2a},$$

where AH is the altitude perpendicular to the side BC. Note that the last formula can be written

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

that is, the Law of Cosines. When the sides b and c and the angle A are known, he presented the following formula to calculate the side a, which is another way of obtaining the Law of Cosines:

$$a^2 = (b \pm c\cos A)^2 + c^2 \sin^2 A,$$

where we have $+2bc\cos A$, if the angle A is obtuse. Continuing the discussion of scalene triangles, he also obtained

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

which is the so–called Law of Sines. At the conclusion of his discussion of scalene triangles, Kashani presented the formulas

$$r_3 = \frac{bc\sin A}{a+b+c},$$

and

$$S_3 = \frac{r_3(a+b+c)}{2}.$$

He claimed that the formula

$$r_3 = \frac{bc\sin A}{a+b+c},$$

was his own discovery. As A. Qurbani pointed out, it seems strange that Kashani did not compare the above two relations to obtain the very useful formula

$$S_3 = \frac{1}{2}bc\sin A.$$

Concerning regular polyhedra, Kashani studied not only the five famous platonic solids (regular tetrahedron, cube, regular octagon, regular dodecahedron, and regular icosahedron) but also two semi–regular polyhedra as well. One was a semi–regular polyhedron with fourteen faces, where eight faces were equilateral triangles, and the other six faces were squares. The length of each edge was equal to the radius of the circumsphere of this semi–regular polyhedron. The other semi–regular polyhedron had thirty two faces, where twenty faces were equilateral triangles and the other twelve faces were pentagons. For each of the above seven solids, Kashani calculated the edge of the solid in terms of the radius of its circumsphere and vice versa. He also calculated the volume of each solid. However, his calculations were not in reduced forms. For instance, he calculated the volume of a regular tetrahedron in terms of the radius R_4 of its circumsphere, without simplifying it, as

$$V_4 = \frac{2}{9}(2R_4)\left(\frac{1}{2}\right)\sqrt{\frac{2}{3}(2R_4)^2}\sqrt{\frac{1}{2}(2R_4)^2};$$

this simplifies to

$$V_4 = \frac{8\sqrt{3}}{27}R_4^3.$$

Kashani calculated x_{12} and x_{20}, the lengths of the edges of a regular dodecahedron and a regular icosahedron in terms of the radii R_{12} and R_{20} of their circumspheres, respectively as follows:

$$x_{12} = \sqrt{5\left(\frac{1}{12}\right)(2R_{12})^2} - \sqrt{\left(\frac{1}{12}\right)(2R_{12})^2}$$

and

$$x_{20} = \sqrt{\left[R_{20} - \sqrt{\left(\frac{1}{20}\right)(2R_{20})^2}\right]^2 + \frac{1}{5}(2R_{20})^2}.$$

The above two formulas reduce to

$$x_{12} = \frac{R_{12}(\sqrt{15} - \sqrt{3})}{3}$$

and

$$x_{20} = \frac{R_{20}}{5}\sqrt{10(5 - \sqrt{5})}.$$

Similarly, he calculated x_{32}, the length of each side of the 32–faced polyhedron described above in terms of the radius R_{32} of its circumsphere as

$$x_{32} = \sqrt{5\frac{(2R_{32})^2}{16}} - \sqrt{\frac{(2R_{32})^2}{16}}$$

which reduces to

$$x_{32} = \frac{R_{32}}{2}(\sqrt{5} - 1).$$

Also, he calculated the diameter of the circumsphere of a regular dodecahedron as

$$(2R_{12})^2 = d^2 = 3\left(x_{12} + \sqrt{x_{12}^2 + \frac{x_{12}^2}{4}} - \frac{x_{12}}{2}\right)^2$$

which reduces to

$$d = \frac{x_{12}}{2}(\sqrt{15} + \sqrt{3}).$$

The eighth chapter is concerned with the calculation of the volumes of some objects from their respective weights. After stating how to obtain the volume of an object from its weight, Kashani presented two tables for 30 substances, including gold, mercury, and lead. In one of these tables, he recorded the weight of water having the same volume as one hundred *methqals* (one *methqal* is approximately 5 grams) of each substance. In the other table he compared the weights of solids occupying the same volume as of one hundred *methqals* of gold. Finally, in the ninth chapter of this book Kashani gave instructions for calculations of volumes and surface areas of vaults, rooms, domes, and other structures. He performed his calculations in both decimal and sexagesimal systems. In dealing with surface areas and volumes of structures, Kashani claimed that his methods were more complete and more efficient than those of his predecessors.

Book V: On the Solution of Problems by Means of Algebra. This book contains four chapters. The first chapter is concerned with *jabr va muqabeleh* and it consists of ten sections. In the first section Kashani defined *jabr, muqabeleh, rad*, and *takmeel* as follows:

Jabr (or algebra). "If on one side or on both sides of an equation there is a negative term, we cancel it and we add a term like it to the other side of the equation."

Muqabeleh. "Cancellation of two equal terms from both sides of an equation is called *muqabeleh.*"

Rad. "If on one side of an equation the coefficient of the highest degree of the unknown is larger than one, then we divide both sides of the equation by this leading coefficient so that the coefficient of the highest degree of the unknown becomes one."

Takmeel. "If on one side of an equation the coefficient of the highest degree of the unknown is less than one, then we multiply both sides of the equation by the reciprocal of this leading coefficient so that the coefficient of the highest degree of the unknown becomes one."

In the second and third sections of the first chapter Kashani presented rules for addition and subtraction of polynomials, and in the fourth section of chapter one he gave a rule for the multiplication of polynomials. Other than the notations used, his rules were similar to modern techniques of polynomial addition, subtraction, and multiplication. In the fifth section of chapter one he discussed division of monomials and polynomials by monomials, and in the sixth he discussed extraction of square roots of monomials and polynomials provided they were perfect squares. He then illustrated his rule of extraction of the square roots by examples including the following three:

$$\sqrt{4x^2 + 20x^3 + 25x^4} = 5x^2 + 2x,$$

$$\sqrt{4x^2 + 20x^3 + 41x^4 + 40x^5 + 16x^6} = 2x + 5x^2 + 4x^3,$$

$$\sqrt{4x^2 + 20x^3 + 41x^4 + 52x^5 + 46x^6 + 24x^7 + 9x^8} = 2x + 5x^2 + 4x^3 + 3x^4.$$

For these three examples, it is clear that Kashani used Equation (1) for the extraction of the square roots.

In the seventh section of chapter one Kashani presented the following six algebraic equations:

$$ax^2 + bx = c,$$

$$ax^2 + c = bx,$$

$$ax^2 = bx + c,$$
$$ax = c,$$

$$ax^2 = bx,$$

$$ax^2 = c.$$

He called the first three *muqtarnat*, and the last three *mufradat*. He discussed their solutions in the eighth section of chapter one. Kashani recognized 95 particular equations of degrees 1, 2, 3, or 4, where 25 of them were of degrees 1, 2, or 3. He stated that these 25 equations had been solved before his time by Sharaf ud–din–e Masudi, but that the remaining 70 equations were unsolved. He wrote that he also solved the aforementioned 25 equations independently and wrote, "I wish I knew whether or not my solutions are simpler than those of Masudi, and whether or not our solutions are compatible." He also claimed that he discovered other problems (equations), and because of lengthy discussions concerning the solutions of these equations he would publish them in another book. Unfortunately, Kashani died before fulfilling this plan.

The reason that Kashani and other mathematicians of his time considered the above quadratic equation as six different types of equations was that they were not using negative numbers. These six equations were well known to Muslim mathematicians as early as third century A. H. L., and they were solved by Omar Khayyam in the fifth century A. H. L. However, it is apparent from Kashani's discussion of these equations that Kashani was not aware of Omar Khayyam's solutions.

In the ninth section of chapter one he solved equations of the form

$$\pm ax^{n+2} \pm bx^{n+1} \pm cx^n = 0$$

by rewriting them as quadratic equations. For example, he reduced the solution of the quintic equation

$$7x^3 = 8x^4 + x^5$$

to the solution of the quadratic equation

$$7 = 8x + x^2,$$

without mentioning the zero as a triple root of the given equation. In the tenth section of the first chapter, Kashani discussed equations of the form $ax^n = bx^m$, where m and n are natural numbers. To solve this equation for $n > m$, he first rewrote it as $x^{n-m} = b/a$, and then as $x = \sqrt[n-m]{b/a}$. However, in his solutions he ignored negative numbers and zero as roots.

The second chapter is concerned with the finding of an unknown by the *hesab–e khataen* ("arithmetic of errors"). Kashani stated that this method is useful only if the given equation can be reduced to a linear equation of the form

$$ax + b = c. \tag{5}$$

In modern notation and terminology the *hesab–e khataen* can be summarized as follows: in (5) we replace x by two arbitrary and different values x_1 and x_2. If either one of these is a solution of (5), then we are done. Otherwise, we have

$$ax_1 + b = c + e_1 \tag{6}$$

and

$$ax_2 + b = c + e_2, \tag{7}$$

where e_1 and e_2 are the first and the second errors, respectively. Now, if we first subtract both sides of equation (5) from corresponding sides of equations (6) and (7), and then divide the corresponding sides of the results we have

$$\frac{x_1 - x}{x_2 - x} = \frac{e_1}{e_2}.$$

From this we deduce that

$$x = \frac{e_2 x_1 - e_1 x_2}{e_2 - e_1}.$$

Since Kashani dealt only with positive numbers, he said if e_1 and e_2 are both negative, then

$$x = \frac{x_1|e_2| - x_2|e_1|}{|e_2| - |e_1|};$$

otherwise (that is, if e_1 and e_2 have opposite signs),

$$x = \frac{x_1|e_2| + x_2|e_1|}{|e_2| + |e_1|}.$$

In the third chapter, Kashani presented fifty formulas, without proofs. Kashani stated that only the seventh, ninth, fifteenth, and sixteenth rules were his own discoveries. Some of these rules are stated here in modern notation.

<u>RULE 7</u>. If a, d, l, and S_n are the first term, the increment, the nth term, and the sum of the first n terms of an arithmetic progression respectively, then

$$l = a + (n-1)d,$$

and

$$S_n = \frac{n(a+l)}{2}.$$

<u>RULE 9</u>. The sum of the first n terms of the geometric progression with ratio 2 is given by

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1.$$

<u>RULE 10</u>.

$$\sum_{k=1}^{n} k(k+1) = 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{2}{3}n\frac{(n+1)(n+2)}{2}.$$

<u>RULE 11</u>.

$$\sum_{k=1}^{n} k(k+1)(k+2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2)$$

$$= \frac{(n+1)(n+2)}{2}\left[\frac{(n+1)(n+2)}{2} - 1\right] = \frac{n(n+1)(n+2)(n+3)}{3}.$$

RULE 12.

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{2n+1}{3}(1+2+3+\cdots+n) = \frac{n(n+1)(2n+1)}{6}.$$

RULE 13.

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$$

$$= \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}.$$

RULE 14.

$$\sum_{k=1}^{n} k^4 = \left(\frac{-1+\sum_{k=1}^{n} k}{5} + \sum_{k=1}^{n} k\right) \sum_{k=1}^{n} k^2.$$

Ibn–e Haytham–e Mesri about four centuries before Kashani obtained a closely related formula for $\sum_{k=1}^{n} k^4$:

$$\sum_{k=1}^{n} k^4 = \left(\frac{n}{5} + \frac{1}{5}\right) n \left(n + \frac{1}{2}\right) \left[(n+1)n - \frac{1}{3}\right].$$

RULE 15.

$$\sum_{k=1}^{n} a^k = a^1 + a^2 + \cdots + a^n = \frac{a \cdot a^n - a}{a-1} = \frac{a(a^n-1)}{a-1} = \frac{a^n - a}{a-1} + a^n,$$

where a is any number except 1. For $a = p/q < 1$, he presented the following formula

$$\sum_{k=1}^{n} \left(\frac{p}{q}\right)^{k} = \frac{(q^n - p^n)p}{(q-p)q^n}.$$

RULE 16. This rule is concerned with the calculation of a^n, when n is large, and we wish to avoid multiplying a by itself n times. Kashani illustrated this rule by the following two examples:

$$5^8 = [(5^2)^2]^2 = [(25)^2]^2 = (625)^2 = 390625,$$

and

$$3^{14} = 3^{2+4+8} = 3^2 \cdot (3^2)^2 \cdot [(3^2)^2]^2 = 9 \cdot 81 \cdot 1561 = 4782969.$$

The fourth chapter of the fifth book is divided into three sections and consists of thirty–nine problems, where Kashani solved them by means of *jabr va muqabeleh, hesab–e khataen*, and *elm–e maftuhat* ("without algebraic equations"). Kashani acknowledged that he obtained some of these problems from *Al–favaed al–bahaeiyyeh*, a book written by Emad ud–din Baqdadi, but Kashani solved them by different methods. The first section contains twenty–five problems involving equations with one or several unknowns of degree one or two, or involving *ma–a–delat–e sayyal* ("flowing equations" or "Diophantine equations"). As an example we reproduce his discussion of the eleventh problem.

We want to partition the number ten into two numbers so that the square of the first number plus the second number is a perfect square. Let x and x' be the two numbers. For $x^2 + x'$ to be a perfect square we must have $x' = 2xy + y^2$ for some y. Thus, we need to solve the equation

$$x + 2xy + y^2 = 10,$$

which implies that

$$x = \frac{10 - y^2}{1 + 2y}. \tag{8}$$

This is a flowing equation which may have more than one solution. If the solutions are restricted to positive integers only, then by (8), y^2 must be less than 10, which means y must be 1, 2, or 3. Second, $10 - y^2$ must be divisible by $1 + 2y$. With these restrictions, $y = 1$ and $x = 3$ is the only acceptable solution to equation (8), and $x' = 2xy + y^2 = 7$, where $3^2 + 7 = 4^2$. If rational numbers are acceptable as well, then there are several solutions including ($x = 6/5$, $x' = 44/5$), and ($x = 1/7$, $x' = 69/7$).

The second and the third sections of chapter four each contains seven problems. All problems in the third section are geometric. For instance, Kashani asks for a point P inside of a given triangle ABC that makes area(PBC) = $(1/2)$area$(PCA) = (1/3)$area(PAB).

Remark. The start of the Islamic calendar is the year 662 A.D., when prophet Muhammad migrated from his hometown of Makkeh to the city of Madineh. Prophet Muhammad's migration is called hejrat (hijra). Nowadays, there are two Islamic calendars in use. One is the lunar calendar and the other is the solar calendar. The lunar calendar is about 354 days long, while the solar calendar is about 365 days long (the same length as the Gregorian calendar). In this paper, "A. H. L." stands for "after hejrat lunar." The twelve months of the Arabic lunar calendar are: Muharram, Safar, Rabi al–awwal, Rabi al–thani, Jamadi al–awwal, Jamadi al–thani, Rajab, Shaban, Ramadan, Shawwal, Dhu–al–qadah, Dhu–al–hijjah.

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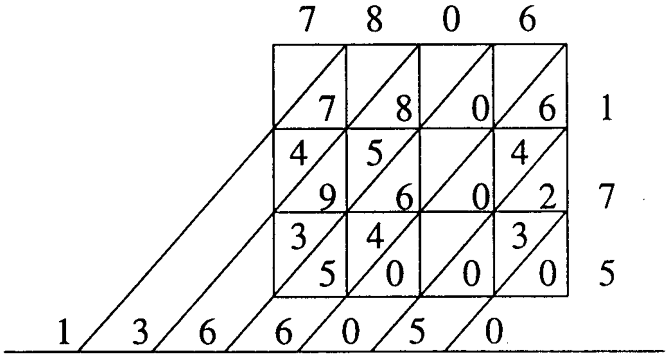

Table 1.

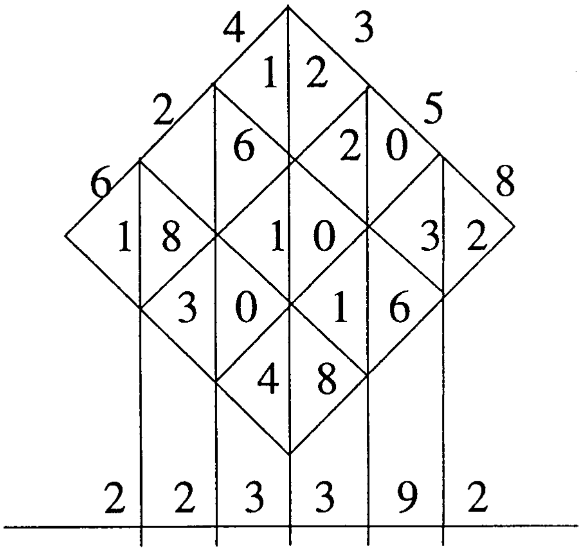

Table 2.