THE POWER INTEGRAL AND THE GEOMETRIC SERIES

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The purpose of this note is to illustrate the importance of some ideas, methods, and techniques of the calculus classics in modern problems of teaching calculus. This approach, combined with the use of technology, provides, in our opinion, a positive contribution to retaining a solid theoretical foundation of the calculus and thus enhancing the success of the learning process. A number of such topics can be offered as student projects in the context of teaching Calculus, History of Mathematics, Differential Equations, Linear Algebra, Real Analysis, etc. These projects will also appeal to the interest of students in other areas of mathematics and at the same time develop their interest to a higher level.

The use of history in teaching mathematics will help convince the students that a college course in the history of mathematics should be primarily a mathematics course, and that a considerable amount of genuine mathematics should be injected in the subject. Such a course will be a study of the development of ideas that shape modern mathematical thinking and mathematicians who contributed those ideas. On the other hand, mathematics did not develop in a vacuum. It has always been an integral part of our life, thinking and culture. It has helped us to uncover the mysteries of nature and create technologies that not only change our world, but also our teaching methods. Therefore, in the teaching of calculus, it is very important to use many ideas and methods of the calculus classics which, in combination with modern technologies, will strengthen and broaden the students liberal education.

One of the examples that we suggest is the use of geometric progressions in the teaching of definite integrals (Fermat's idea).

Pierre Fermat (1601–65) is famous not only for his Last Theorem; he is also known as a founder of the modern theory of numbers and probability theory. He also did much to establish coordinate geometry and invented a number of methods for determining maxima and minima that were later of use to Newton in founding the calculus. Fermat recognized a principle in optics known as Fermat's Law. Fermat is also credited with a method of calculating areas under certain curves by partitioning the basic interval with a sequence of points whose coordinates form a geometric progression.

In this connection, we want to compare Fermat's method with the approach discussed in the note by Mathews [1] for the integral of the power function. In his capsule, Mathews found the integrals of $t^{1/2}$ and $t^{4/3}$ on $0 \leq t \leq x$ by Riemann sums with partition points

$$x_k = \frac{k^2 x}{n^2} \quad \text{and} \quad x_k = \frac{k^3 x}{n^3}, \quad k = 0, 1, \ldots, n$$

respectively, and by using closed formulas for the sums of k and k^2 (integral of $t^{1/2}$) and k^4, k^5, and k^6 (integral of $t^{4/3}$). Incidentally, recursive formulas for the sums of integer powers can be derived by means of differentiation [3]. Furthermore, we would like to observe that the use of these formulas in the calculation of the above integrals may be avoided, and only the knowledge of the limit

$$\lim_{n\to\infty} \frac{1^m + 2^m + 3^m + \cdots + n^m}{n^{m+1}} = \frac{1}{m+1} \tag{1}$$

for particular values of m is essential. In the approach described below this remarkable limit will be obtained in general form as a corollary of the main result.

Our preliminary discussion of Fermat's ingenious method follows the account given in the book *Calculus Gems* by George F. Simmons [2]. However a detailed analysis of the method shows that it is by far more significant in calculus than exhibited in the literature. Therefore, the following related topics are also included in this investigation.

1. Generalization of Fermat's method for the integral of t^m with positive rational m to all real $m > -1$.
2. Derivation of limit (1) by comparing two integer sums for t^m; one with a geometric sequence and the other with an arithmetic sequence of partition points.
3. The integral definition of the logarithmic function.
4. Estimates for the partial sums of the harmonic series.
5. The integral of $\ln t$.

Consider the power function $f(t) = t^m$ where $m > -1$.To find its integral on $0 \le t \le x$, Fermat divides this interval by the geometric sequence of points (moving from right to left)

$$x, xr, xr^2, xr^3, \ldots, xr^{k-1}, \ldots$$

where $0 < r < 1$ (Figure 1), and considers the following integral sum:

$$\begin{aligned} S_1 &= x^m (x - xr) + (xr)^m (xr - xr)^2 + \left(xr^2\right)^m \left(xr^2 - xr^3\right) + \cdots \\ &= x^{m+1} (1 - r) \sum_{k=0}^{\infty} r^{(m+1)k} = x^{m+1} \frac{1 - r}{1 - r^{m+1}}. \end{aligned} \tag{2}$$

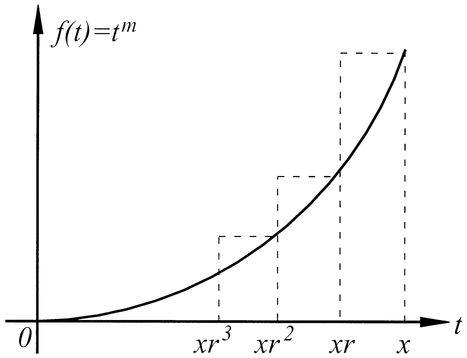

Figure 1.

We remark that this expression for the integral sum looks unusual because it is given in the form of an infinite series. However, instead of the interval $[0, x]$ we can take an interval $[\Delta, x]$, with $0 < \Delta < x$, then write a finite Riemann sum and make Δ approach 0. Now, the integral of t^m on $[0, x]$ is the limit of (2) as $r \to 1$. If $m \geq 0$ is an integer, then

$$1 - r^{m+1} = (1 - r)\left(1 + r + r^2 + \cdots + r^m\right)$$

and

$$S_1 = \frac{x^{m+1}}{1 + r + r^2 + \cdots + r^m}$$

So,

$$\int_0^x t^m dt = \lim_{r \to 1} S_1 = \frac{x^{m+1}}{m+1}. \tag{3}$$

The same result follows from (2) for any real $m > -1$ by the application of L'Hôpital's rule.

Moreover, let us compare the integral sum (2) with a sum obtained by dividing the interval $[0,x]$ with the arithmetic sequence $0,h,2h,\dots,nh=x$. We write the integral sum

$$
\begin{aligned}
S_2 &= h^m h + (2h)^m h + \cdots + (nh)^m h \\
&= h^{m+1}\left(1^m + 2^m + 3^m + \cdots + n^m\right) \\
&= \frac{1^m + 2^m + 3^m + \cdots + n^m}{n^{m+1}} x^{m+1}.
\end{aligned}
$$

Since for $n \to \infty$ this sum tends to the same limit as (2), which is $x^{m+1}/(m+1)$, then the remarkable limit (1) follows from here for any real $m > -1$.

For the integral

$$\int_1^x t^m dt = \int_0^x t^m dt - \int_0^1 t^m dt$$

it follows from (3) that

$$\int_1^x t^m dt = \frac{x^{m+1}-1}{m+1},$$

and letting $m \to -1$ in this equation gives

$$\int_1^x \frac{dt}{t} = \lim_{m\to -1} \frac{x^{m+1}-1}{m+1} = \lim_{m\to -1} \frac{x^{m+1}\ln x}{1}$$

and generates the integral definition of the logarithm

$$\int_1^t \frac{dt}{t} = \ln x, \quad x > 0.$$

An immediate consequence of this formula is the conclusion on the divergence of the harmonic series. In fact, by the Mean Value Theorem we have

$$\int_k^{k+1} \frac{dt}{t} = \ln(k+1) - \ln k = \frac{1}{k+c},$$

where $0 < c < 1$, and therefore,

$$\frac{1}{k+1} < \ln(k+1) - \ln k < \frac{1}{k}.$$

Adding these inequalities for $k = 1, 2, \ldots, n$ gives

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} < \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{4}$$

and clearly shows that the harmonic series diverges since its nth partial sum

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is unbounded being greater than $\ln(n+1)$.

Furthermore, this discussion leads to a remarkable estimate for the growth rate of the partial sums of the harmonic series. Indeed, subtracting from H_n each part of the inequalities (4), it is seen that

$$0 < H_n - \ln(n+1) < 1 - \frac{1}{n+1} < 1.$$

From here, we conclude that the variable $H_n - \ln(n+1)$ has a finite limit as $n \to \infty$, since it is increasing and bounded from above. This famous limit,

$$\lim_{n\to\infty} [H_n - \ln(n+1)] = \gamma,$$

is known as the Euler–Mascheroni constant. In other words,

$$H_n = \ln(n+1) + \gamma + \beta_n,$$

where $\beta_n \to 0$ as $n \to \infty$. Although the number γ can be evaluated with any precision ($\gamma = 0.57721566\ldots$), its arithmetic nature remains the greatest enigma, and nobody has shown that it cannot be rational.

Euler used the divergence of the harmonic series to prove that the number of primes is infinite. Assume that the number of primes is finite and denote all primes by $p_1, p_2, p_3, \ldots, p_N$. Let us take the following fractions:

$$\frac{1}{1-\frac{1}{p_1}}, \quad \frac{1}{1-\frac{1}{p_2}}, \quad \ldots, \frac{1}{1-\frac{1}{p_N}}.$$

Each of these terms can be written as a geometric series.

$$\frac{1}{1-\frac{1}{p_i}} = 1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \cdots, \quad i = 1, 2, \ldots, N.$$

Multiplying both sides of these N equalities, we obtain

$$\begin{aligned}\prod_{i=1}^{N} \frac{1}{1-\frac{1}{p_i}} &= \left(1 + \frac{1}{p_1} + \frac{1}{p_2} + \cdots\right) \cdot \cdots \cdot \left(1 + \frac{1}{p_N} + \frac{1}{p_N^2} + \cdots\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .\end{aligned}$$

Hence, the harmonic series is convergent, a contradiction.

For the integral of $\ln t$ between 0 and x, Fermat's method generates the integral sum

$$\begin{aligned}S &= (x - xr)\ln x + \left(xr - xr^2\right)\ln(xr) + \left(xr^2 - xr^3\right)\ln\left(xr^2\right) + \cdots \\ &= x(1-r)\left[\ln x + r\ln(xr) + r^2\ln\left(xr^2\right) + \cdots\right],\end{aligned}$$

which is transformed into

$$S = x(1-r)\left(1 + r + r^2 + \cdots\right)\ln x + x(1-r)\,r\left(1 + 2r + 3r^2 + 4r^3 + \cdots\right)\ln r,$$

or

$$S = x\ln x + xr\frac{\ln r}{1-r}.$$

Hence,

$$\int_0^x \ln t dt = \lim_{r \to 1} S = x \ln x - x. \tag{5}$$

We may only guess how Fermat had arrived at his method. However, the substitution $t = e^s$ changes the integral of $e^{(m+1)s}ds$ to the power integral (3). Therefore, if the partition points for the integral of the exponential function form an arithmetic sequence, the partition points for the power integral create a geometric sequence.

References

1. J. H. Mathews, "The Integral of $x^{1/2}$, etc.," *College Mathematics Journal*, 25 (1994), 142–144.
2. G. F. Simmons, *Calculus Gems*, McGraw Hill, New York, 1992.
3. J. Wiener, "A Calculus Exercise for the Sums of Integer Powers," *Mathematics Magazine*, 65 (1992), 249–251.

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