SYMMETRIC PYTHAGOREAN TRIPLE PRESERVING MATRICES

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Abstract. A Pythagorean Triple Preserving Matrix (PTPM) is a 3×3 matrix such that if it is multiplied by a Pythagorean Triple, the result is also a Pythagorean Triple. Necessary and sufficient conditions for a Pythagorean Triple Preserving Matrix to be symmetric are given. Monoids of Symmetric Pythagorean Triple Preserving Matrices (SPTPM) with positive integer entries will be developed, and to ensure that the set is closed under matrix multiplication, the focus will be on finding commutative SPTPM's.

1. Introduction. A triple of positive integers (a, b, c) is defined to be a Pythagorean Triple if it satisfies the equation $a^2 + b^2 = c^2$. Moreover, (a, b, c) is said to be a Primitive Pythagorean Triple (PPT) if $\gcd(a, b, c) = 1$ [3].

It is known [1] that all Primitive Pythagorean Triples (a, b, c) are of the form $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ where m, n satisfy the conditions listed below:

I.1 m, n are positive integers.
I.2 $m > n$.
I.3 $\gcd(m, n) = 1$.
I.4 $m \not\equiv n \pmod 2$.

It should be noted that a triple $(m^2 - n^2, 2mn, m^2 + n^2)$ satisfies the Pythagorean Theorem for every value of m and n. However, only those that satisfy conditions I.1 to I.4 above are called PPT's. Also, this form implies that if (a, b, c) is a Primitive Pythagorean Triple, then b must be even.

A 3×3 matrix A with integer entries is called a Pythagorean Triple Preserving Matrix if whenever (a, b, c) is a PT, then $(d, e, f) = (a, b, c)A$ is also a PT [1]. For example, let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

One can easily see that the Pythagorean Triple $(3,4,5)$ multiplied by A gives $(3,4,5)A = (21,20,29)$, which is yet another Pythagorean Triple [4]. In two recent papers by Palmer, Ahuja, and Tikoo [1,2], Pythagorean Triple Preserving Matrices have been identified and constructed.

The following lemma and theorem are proven by Palmer, Ahuja, and Tikoo [1].

Lemma 1. Let A denote the matrix

$$A = \begin{bmatrix} \frac{(r^2-t^2)-(s^2-u^2)}{2} & rs-tu & \frac{(r^2-t^2)+(s^2-u^2)}{2} \\ rt-su & ru+st & rt+su \\ \frac{(r^2+t^2)-(s^2+u^2)}{2} & rs+tu & \frac{(r^2+t^2)+(s^2+u^2)}{2} \end{bmatrix}. \tag{1}$$

Then $(m^2-n^2, 2mn, m^2+n^2)A = (M^2-N^2, 2MN, M^2+N^2)$, where $M = mr+nt$, $N = ms+nu$. The relation between (m,n) and (M,N) can be expressed by the matrix equation

$$(m,n)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = (M,N). \tag{2}$$

Theorem 1. A 3×3 matrix is a PTPM if and only if it is of form A.

The authors in [1] further show that the values of r, s, t, u should be restricted in the following ways:

R-1. r, s, t, u are integers with $r, s > 0$.
R-2. $r+t \geq s+u \geq 0$. (Later in this paper, it will be further assumed that $r > u$.)
R-3. $ru - st = \pm 1$.
R-4. $r+s \equiv t+u \equiv 1 \pmod 2$.

These restrictions are sufficient to assure that the second triple $(M^2 - N^2, 2MN, M^2+N^2)$ is a PPT whenever $(m^2-n^2, 2mn, m^2+n^2)$ is a PPT. It is also shown in [1] that $\det A = (ru-st)^3$, so since $(\pm 1)^3 = \pm 1$, only PTPM's with determinant ± 1 will be considered.

A semigroup is a nonempty set G together with an associative binary operation; a monoid is a semigroup G which contains a (two–sided) identity element $e \in G$ such that $ae = ea = a$ for all $a \in G$ [6]. Therefore since (1) the identity is obviously a PTPM, (2) the set of PTPM's is easily seen to be closed under the operation of matrix multiplication, and (3) matrix multiplication is associative, it is known that the set of PTPM's forms a monoid.

A square matrix S is said to be symmetric if $S = S^T$, where S^T denotes the transpose of S. The set of all Symmetric Pythagorean Triple Preserving Matrices is not closed under the operation of matrix multiplication since the product of two symmetric matrices may or may not be symmetric. It can be easily shown, however, that the product of two symmetric matrices is symmetric if and only if the two matrices commute.

It is known for symmetric matrices [7,8] that any two eigenvectors from different eigenspaces are orthogonal. Furthermore, an $(n \times n)$ matrix A is orthogonally diagonalizable if and only if A is symmetric, and a matrix is known to be diagonalizable if and only if it possess a set of n linearly independent eigenvectors. An $n \times n$ matrix A is defined to be semisimple if it has a total of n linearly independent eigenvectors [5], so every symmetric matrix is semisimple. The following theorem [5] will then guarantee when the product of two SPTPM's is symmetric.

Theorem 2. If two matrices are semisimple, then they commute if and only if they have a complete set of eigenvectors in common.

2. Conditions That Guarantee Symmetry.

Theorem 3. Let

$$A = \begin{bmatrix} \frac{(r^2-t^2)-(s^2-u^2)}{2} & rs - tu & \frac{(r^2-t^2)+(s^2-u^2)}{2} \\ rt - su & ru + st & rt + su \\ \frac{(r^2+t^2)-(s^2+u^2)}{2} & rs + tu & \frac{(r^2+t^2)+(s^2+u^2)}{2} \end{bmatrix}.$$

Then A is symmetric if and only if $s = t$.

The proof follows in a straightforward manner.

Remark. Note that if $s = t$, then the matrix

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

in equation (2) is symmetric as well.

The symmetric PTPM, for the remainder of this paper, will be denoted J, where J has the form

$$J = \begin{bmatrix} \frac{r^2-2s^2+u^2}{2} & rs-su & \frac{r^2-u^2}{2} \\ rs-su & ru+s^2 & rs+su \\ \frac{r^2-u^2}{2} & rs+su & \frac{r^2+2s^2+u^2}{2} \end{bmatrix}.$$

The symmetry condition $s = t$ when applied to the determinant condition, R-3 ($ru - st = \pm 1$), produces the two equations, $ru - s^2 = 1$ and $ru - s^2 = -1$. The first of these equations can be rewritten $ru = s^2 + 1$, but this rewritten form does not have any obvious solutions. The second equation, though, when rewritten $ru = s^2 - 1 = (s+1)(s-1)$ produces the natural solutions $r = s + 1$, $u = s - 1$. Substituting these into (1) generates symmetric PTPM's of the form

$$C = \begin{bmatrix} 1 & 2s & 2s \\ 2s & 2s^2-1 & 2s^2 \\ 2s & 2s^2 & 2s^2+1 \end{bmatrix},$$

where $\det C = (ru - s^2) = -1$. The example

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

illustrated earlier in this paper is generated by $r = 2$, $s = t = 1$, $u = 0$.

It is a routine matter to show that J is a symmetric PTPM of form C if and only if $r = s + 1$ and $u = s - 1$. The significance of matrices of form C will be illustrated in later sections when monoids of symmetric PTPM's are studied.

3. Eigenvalues and Eigenvectors. All SPTPM's are of the form

$$J = \begin{bmatrix} \frac{r^2-2s^2+u^2}{2} & rs-su & \frac{r^2-u^2}{2} \\ rs-su & ru+s^2 & rs+su \\ \frac{r^2-u^2}{2} & rs+su & \frac{r^2+2s^2+u^2}{2} \end{bmatrix},$$

where $r, s > 0$, $ru - s^2 = \pm 1$. The condition R-2 ($r > u$) guarantees the positive nature of the elements and the condition R-4 ($r+s \equiv t+u \equiv 1 \pmod 2$) guarantees that the entries are integers. Through the use of computer software, the eigenvalues λ_i and eigenvectors x_i of J were found:

$$\lambda_1 = ru - s^2 \qquad x_1 = \begin{bmatrix} \frac{2s}{u-r} \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{r^2+2s^2+u^2}{2} + \frac{|r+u|\sqrt{r^2-2ru+4s^2+u^2}}{2} \qquad x_2 = \begin{bmatrix} 1 \\ \frac{2s}{r-u} \\ \frac{\sqrt{r^2-2ru+4s^2+u^2}}{r-u} \end{bmatrix}.$$

$$\lambda_3 = \frac{r^2+2s^2+u^2}{2} + \frac{|r+u|\sqrt{r^2-2ru+4s^2+u^2}}{2} \qquad x_3 = \begin{bmatrix} 1 \\ \frac{2s}{r-u} \\ \frac{\sqrt{r^2-2ru+4s^2+u^2}}{u-r} \end{bmatrix}$$

4. Conditions for SPTPM's to Commute.

Theorem 4. Let J_1 and J_2 be SPTPM's generated by r_1, s_1, and u_1, and r_2, s_2, and u_2, respectively. Then J_1 and J_2 commute if and only if

$$\frac{s_1}{r_1 - u_1} = \frac{s_2}{r_2 - u_2}.$$

Proof. Assume the semisimple matrices J_1 and J_2 commute. Let the eigenvectors be as follows:

$$J_1: \quad x_1 = \begin{bmatrix} \frac{2s_1}{u_1-r_1} \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ \frac{2s_1}{r_1-u_1} \\ \frac{\sqrt{r_1^2-2r_1u_1+4s_1^2+u_1^2}}{r_1-u_1} \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ \frac{2s_1}{r_1-u_1} \\ \frac{\sqrt{r_1^2-2r_1u_1+4s_1^2+u_1^2}}{u_1-r_1} \end{bmatrix}$$

and

$$J_2: \quad y_1 = \begin{bmatrix} \frac{2s_2}{u_2-r_2} \\ 1 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 \\ \frac{2s_2}{r_2-u_2} \\ \frac{\sqrt{r_2^2-2r_2u_2+4s_2^2+u_2^2}}{r_2-u_2} \end{bmatrix}, \quad y_3 = \begin{bmatrix} 1 \\ \frac{2s_2}{r_2-u_2} \\ \frac{\sqrt{r_2^2-2r_2u_2+4s_2^2+u_2^2}}{u_2-r_2} \end{bmatrix}.$$

Since J_1 and J_2 must have a complete set of common eigenvectors by Theorem 2, the pairing $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, is the only possibility. This results in the following conditions:

$$\frac{2s_1}{r_1 - u_1} = \frac{2s_2}{r_2 - u_2}, \tag{3}$$

and

$$\frac{\sqrt{r_1^2 - 2r_1u_1 + 4s_1^2 + u_1^2}}{r_1 - u_1} = \frac{\sqrt{r_2^2 - 2r_2u_2 + 4s_2^2 + u_2^2}}{r_2 - u_2}. \tag{4}$$

It can be shown that (3) and (4) are equivalent. (Assume $r > u$.) This implies that when the eigenvectors are in common, then

$$\frac{s_1}{r_1 - u_1} = \frac{s_2}{r_2 - u_2}.$$

Conversely, assume

$$\frac{s_1}{r_1 - u_1} = \frac{s_2}{r_2 - u_2}$$

and let the eigenvectors of J_1 and J_2 be the same as above. Since equations (3) and (4) are equivalent, the eigenvectors of J_1 and J_2 are in common by the pairing $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, and the matrices commute by Theorem 2.

Corollary 1. Let C_1 be a matrix of form C generated by $s = k$ and let J_1 be a matrix of form J generated by r_1, s_1, and u_1. C_1 and J_1 commute if and only if

$$k = \frac{2s_1}{r_1 - u_1}.$$

Corollary 2. If C_1 and C_2 are two SPTPM's of form C, then C_1 and C_2 commute if and only if $C_1 = C_2$.

Theorem 5. If J_1 and C_1 are SPTPM's of form J and C, respectively, then J_1 and C_1 commute if and only if $J_1 = C_1^n$, $n \geq 0$. ($C_1^0 = I_3$, the 3 by 3 identity matrix.)

Proof. Let $J_1 = C_1^n$, $n \geq 0$. Then $J_1 \cdot C_1 = C_1^n \cdot C_1 = C_1^{n+1} = C_1 \cdot C_1^n = C_1 \cdot J_1$, and the matrices commute.

Now assume J_1 and C_1 commute, where

$$C_1 = \begin{bmatrix} 1 & 2k & 2k \\ 2k & 2k^2-1 & 2k^2 \\ 2k & 2k^2 & 2k^2+1 \end{bmatrix},$$

k a positive integer, and J_1 is an SPTPM generated by r_1, s_1, u_1. From condition R-3,

$$r_1 u_1 - s_1^2 = \pm 1, \tag{5}$$

and from Corollary 1,

$$k = \frac{2s_1}{r_1 - u_1}. \tag{6}$$

Combining equations (5) and (6),

$$\left(\frac{2s_1 + ku_1}{k}\right) u_1 - s_1^2 = \pm 1$$

$$\begin{aligned} &ku_1^2 + 2s_1u_1 - ks_2 = \pm k \\ &ku_1^2 + 2s_1u_1 - ks_2 \pm k = 0. \end{aligned}$$

The quadratic formula will then give

$$u_1 = \frac{-s_1 + \sqrt{s_1^2(1+k^2) \pm k^2}}{k}. \tag{7}$$

Similarly,

$$r_1 = \frac{s_1 + \sqrt{s_1^2(1+k^2) \pm k^2}}{k}. \tag{8}$$

The minus signs before the radicals in u_1 and r_1 from the quadratic formula are eliminated to guarantee r_1 is positive, and so that equation (6) is satisfied.

From condition R-4, $r_1 - u_1$ must be even, and equation (6) implies

$$\frac{r_1 - u_1}{2} = \frac{s_1}{k},$$

so that k divides s_1. Let $s_1 = q \cdot k$. Then $\sqrt{s_1^2(k^2+1) \pm k^2} = k\sqrt{q^2(k^2+1) \pm 1}$. If $p^2 = q^2(k^2+1) \pm 1$, p an integer, then

$$p^2 - q^2(k^2+1) = \pm 1, \tag{9}$$

is a form of Pell's equation, and solutions (p, q) can be found by methods outlined in number theory books [3,9]. It can be shown that all positive integer solutions of (9) are of the form (p_n, q_n), $n \geq 1$, where $p_0 = 1$, $p_1 = k, \ldots, p_n = 2k \cdot p_{n-1} + p_{n-2}$, and $q_0 = 0$, $q_1 = 1, \ldots, q_n = 2k \cdot q_{n-1} + q_{n-2}$. Furthermore, $p_n^2 - q_n^2(k^2+1) = (-1)^n$.

Then $u_1 = -q_n + \sqrt{q_n^2(k^2+1) + (-1)^n}$, $r_1 = q_n + \sqrt{q_n^2(k^2+1) + (-1)^n}$, and $s_1 = k \cdot q_n$ for each $n \geq 1$. This implies that the matrix J_1 which commutes with C_1 will be one of the sequence of matrices determined by a triple (r_1, s_1, u_1) generated by values of q_n.

If $q_0 = 0$ for $n = 0$, then $r_1 = 1$, $s_1 = 0$, $u_1 = 1$, and $J_1 = I_3 = C_1^0$. For $n = 1$, $q_1 = 1$, so that $r_1 = k + 1$, $s_1 = k$, and $u_1 = k - 1$. The matrix J_1 determined by these values is C_1. Now for $n = 2$, $q_1 = 2k$, which makes $r_1 = 2k^2 + 2k + 1$, $s_1 = 2k^2$, and $u_1 = 2k^2 - 2k + 1$. These values will produce $J_1 = C_1^2$. It can be shown by math induction that C_1^n is generated by $s = q_n \cdot k$, $r = q_n + \sqrt{q_n^2(k^2+1) + (-1)^n}$, and $u = -q_n + \sqrt{q_n^2(k^2+1) + (-1)^n}$ so that the pattern will continue and clearly imply that J_1 is a power of C_1.

<u>Corollary 3</u>. Let J_1 be a matrix of form J generated by r_1, s_1, u_1. J_1 is a power of a matrix of form C if and only if

$$k = \frac{2s_1}{r_1 - u_1}$$

is an integer. In this case, J_1 is a power of the matrix of form C generated by the integer k.

As was seen in Theorem 4, two matrices J_1 and J_2 of form J commute if and only if

$$\frac{s_1}{r_1 - u_1} = \frac{s_2}{r_2 - u_2}.$$

If the ratio $2s/(r-u)$ for each of the matrices J_1 and J_2 equals an integer, then J_1 and J_2 are powers of a matrix of form C and obviously commute. The question remains, though, for what fractions p/q do there exist commuting matrices for which

$$\frac{2s}{r-u} = \frac{p}{q}.$$

Consider the rewritten form of the condition of Theorem 4

$$s_2 = \frac{s_1(r_2 - u_2)}{r_1 - u_1}. \tag{10}$$

The r, s, and u values in the table below for matrices J_1 and J_2 were computer–derived values to satisfy equation (10) and generate matrices that commute.

J_1 matrix				J_2 matrix			
r_1	s_1	u_1	$\frac{2s_1}{r_1-u_1}$	r_2	s_2	u_2	$\frac{2s_2}{r_2-u_2}$
5	2	1	$\frac{1}{1}$	12	5	2	$\frac{1}{1}$
17	4	1	$\frac{1}{2}$	72	17	4	$\frac{1}{2}$
37	6	1	$\frac{1}{3}$	228	37	6	$\frac{1}{3}$
65	8	1	$\frac{1}{4}$	528	65	8	$\frac{1}{4}$
101	10	1	$\frac{1}{5}$	1020	101	10	$\frac{1}{5}$

Notice that the fractions are of the form $1/n$, and the values of r, s, and u seem to be forming a sequence for each $1/n$. For each fraction p/q in lowest terms not equal to an integer, it is possible that there will exist a set of commuting matrices of form J such that for each J in the set,

$$\frac{2s}{r-u} = \frac{p}{q}.$$

It is conjectured, however, considering computer–generated examples similar to those listed above, that only fractions of the form $1/n$ admit a set of commuting matrices.

5. Monoids of Commuting Matrices. The earlier theorems established conditions for Symmetric Pythagorean Triple Preserving matrices to commute, thereby producing products that are symmetric. The goal was to find closed sets of SPTPM's under matrix multiplication. From Theorem 5 and Corollary 3, it is seen that the value of the ratio $2s/(r-u)$ will generate different monoids. The identity matrix must be in each monoid, but since the values $r = 1$, $s = 0$, $u = 1$ produce the identity, the value of the ratio $2s/(r-u)$ for the identity is the indeterminate $0/0$. The identity matrix, which is a SPTPM, will thus be considered an element in each monoid.

Let k be an integer. Let C_k denote the matrix of form C generated by the integer k. The set of all nonnegative powers of C_k (the identity taken as C_k^0) will form a closed set under matrix multiplication and thus be a monoid. Theorem 5 gives the method for completely describing each matrix in this monoid. Thus, for each integer k there exists a monoid of SPTPM's such that the ratio $2s/(r-u)$ can be considered to equal k for all matrices in the monoid.

Next consider monoids corresponding to the fraction $1/2$. Let

$$J_0 = \begin{bmatrix} 7 & 4 & 8 \\ 4 & 1 & 4 \\ 8 & 4 & 9 \end{bmatrix},$$

$r = 4$, $s = 1$, $u = 0$; and let

$$J_1 = \begin{bmatrix} 129 & 64 & 144 \\ 64 & 33 & 72 \\ 144 & 72 & 161 \end{bmatrix},$$

$r = 17$, $s = 4$, $u = 1$. If J_2 is generated by $r = 72$, $s = 17$, $u = 4$, J_3 by $r = 305$, $s = 72$, $u = 17$, J_4 by $r = 1292$, $s = 305$, $u = 72$, and J_5 by $r = 5473$, $s = 1292$, $u = 305$, then for each of these,

$$\frac{2s}{r-u} = \frac{1}{2}.$$

$J_1 = J_0^2$, $J_2 = J_0^3$, $J_3 = J_0^4 = J_1^2$, $J_4 = J_0^5$, and $J_5 = J_0^6 = J_1^3 = J_2^2$. The powers of J_0 form a monoid, and since these powers will commute,

$$\frac{2s}{r-u} = \frac{1}{2}.$$

It can be shown that the powers of J_0 will form the complete monoid for $1/2$.

Notice for each J_i in the paragraph above, the s and u values of J_i are the r and s values, respectively, of J_{i-1}, for each J_i, $r = 4s + u$, or

$$\frac{2s}{r-u} = \frac{1}{2}.$$

The monoid for $1/2$ seems to also be determined by the sequence $1, 0, 1, 4, 17, 72, 305, 1292, 5473, \ldots$, where the u, s, and r values ($r = 4s + u$) for a particular matrix in the monoid will be three consecutive elements of the sequence. ($1, 0, 1$ determines the identity, or J_0^0.) Again, it can be shown that the sequence listed above will generate all matrices in the monoid for $1/2$.

In the general case, matrices in the monoid for the fraction $1/n$ will have u, s, and r values that come from the sequence $1, 0, 1, 2n, 4n^2 + 1, 8n^3 + 4n, 16n^4 + 12n^2 + 1, \ldots$, where the elements of this sequence satisfy the recurrence relationship $a_j = 2n \cdot a_{j-1} + a_{j-2}$. If $u = a_{j-2}$, $s = a_{j-1}$, and $r = a_j = 2n \cdot a_{j-1} + a_{j-2}$, then it can be easily seen that

$$\frac{2s}{r-u} = \frac{1}{n}.$$

By a method similar to the proof of Theorem 5, it can be conclusively shown that the powers of an SPTPM J_0, with $r = 2n$, $s = 1$, and $u = 0$, will generate all matrices in the monoid for $1/n$, and further, that the sequence listed above will generate all the matrices in the monoid.

6. Conclusion. The main focus of this paper was to find sets of Symmetric Pythagorean Triple Preserving Matrices (SPTPM) that are closed under matrix multiplication. The general form A of a Pythagorean Triple Preserving Matrix was first revised to develop a general form J for SPTPM's with parameters r, s, and u. Developing closed sets of SPTPM's involved finding products of symmetric matrices that would again be symmetric, but this implied the products of these symmetric matrices must be commutative. Since this is known to occur when the two matrices have a set of eigenvectors in common, the condition

$$\frac{s_1}{r_1 - u_1} = \frac{s_2}{r_2 - u_2},$$

that will guarantee when two SPTPM's commute, was established by equating the eigenvectors of two SPTPM's J_1 and J_2 generated by r_1, s_1, u_1 and r_2, s_2 u_2, respectively. From this condition then, it was determined that two types of monoids of commutative SPTPM's are now known to exist, and each matrix in a particular monoid satisfies either the condition

$$\frac{2s}{r - u} - k,$$

k an integer, or

$$\frac{2s}{r - u} = \frac{1}{n},$$

n an integer.

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