A REPRESENTATION AND SOME PROPERTIES FOR k–FIBONACCI SEQUENCES

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Abstract. The k–Fibonacci sequence $\{g_n^{(k)}\}$ is defined as:

$$g_1^{(k)} = \ldots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

In this paper, we give a combinatorial representation of $g_n^{(k)}$ and give some properties for k–Fibonacci sequence.

1. Introduction. The well–known Fibonacci sequence $\{F_n\}$ is defined as:

$$F_1 = F_2 = 1 \quad \text{and, for} \quad n > 2, \quad F_n = F_{n-1} + F_{n-2}.$$

We call F_n the nth Fibonacci number. The Fibonacci sequence is

$$(F_0 := 0), 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

Now, we consider the generalization of the Fibonacci sequence, which is called the k–Fibonacci sequence for the positive integer $k \geq 2$. The k–Fibonacci sequence $\{g_n^{(k)}\}$ is defined as:

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the nth k–Fibonacci number. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence, $\{F_n\}$, and if $k = 4$, then $g_1^{(4)} = g_2^{(4)} = 0$, $g_3^{(4)} = g_4^{(4)} = 1$, and then the 4-Fibonacci sequence is

$$0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, \ldots.$$

Let I_{k-1} be the identity matrix of order $k-1$ and let E be an $1 \times (k-1)$ matrix whose entries are ones. For any $k \geq 2$, the fundamental recurrence relation

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$$

can be defined by the vector recurrence relation

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k \begin{bmatrix} g_{n}^{(k)} \\ g_{n+1}^{(k)} \\ \vdots \\ g_{n+k-1}^{(k)} \end{bmatrix} \tag{1.1}$$

where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}_{k\times k}. \tag{1.2}$$

The matrix Q_k is said to be the k–Fibonacci matrix. By applying (1.1), we have

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k^n \begin{bmatrix} g_{1}^{(k)} \\ g_{2}^{(k)} \\ \vdots \\ g_{k}^{(k)} \end{bmatrix}.$$

Let $\{g_n^{(k)}\}$ be a k–Fibonacci sequence, and let

$$G_k = (g_1,\ g_2,\ g_3,\ \ldots),\quad g_i = g_{i+k-2}^{(k)},\quad i = 1,\ 2,\ \ldots\ ,$$

and if $i \leq 0$, then $g_i = 0$.

For example, if $k = 2$, then $G_2 = (1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \ldots)$. And if $k = 4$, then $G_4 = (1,\ 1,\ 2,\ 4,\ 8,\ 15,\ 29,\ 56,\ 108, \ldots)$.

In [3], the author considered the completeness on $\{g_n^{(k)}\}$ and gave a representation for the recurrence relation $g_n^{(k)}$. In [4], the authors found a relationship between the k–Fibonacci number $g_n^{(k)}$ and the number of 1-factors of a bipartite graph, and in [5], the authors considered the eigenvalues of k–Fibonacci matrix Q_k and gave some interesting examples in combinatorics and probability with respect to the k–Fibonacci sequences.

In this paper, we give a combinatorial representation of $g_n^{(k)}$ and introduce some properties for k–Fibonacci sequences.

2. Combinatorial representation of $\mathbf{g_n}$. In this section, we give a representation for the nth k–Fibonacci number by using the generating function $G_k(x)$.

We can easily find the characteristic polynomial, $x^k - x^{k-1} - \cdots - x - 1$, of the k–Fibonacci matrix Q_k. It follows that all of the eigenvalues of Q_k satisfy

$$x^k = x^{k-1} + x^{k-2} + \cdots + x + 1.$$

And we can find the following fact in [5]:

$$\begin{aligned} x^n = g_{n-k+2}x^{k-1} &+ (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+3})x^{k-2} \\ &+ (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+4})x^{k-3} \\ &+ \cdots + (g_{n-k+1} + g_{n-k})x + g_{n-k+1}. \end{aligned} \tag{2.1}$$

Let

$$G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots .$$

Then

$$G_k(x) - xG_k(x) - x^2G_k(x) - \cdots - x^kG_k(x) = (1 - x - x^2 - \cdots - x^k)G_k(x).$$

Using equation (2.1), we have

$$(1 - x - x^2 - \cdots - x^k)G_k(x) = g_1 = 1.$$

Thus,

$$G_k(x) = (1 - x - x^2 - \cdots - x^k)^{-1}$$

for $0 \le x + x^2 + \cdots + x^k < 1$.

Let $f_k(x) = x + x^2 + \cdots + x^k$. Then $0 \le f_k(x) < 1$ and we have the following lemma.

Lemma 2.1. For positive integers p and n, the coefficient of x^n in $(f_k(x))^p$ is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \quad \frac{n}{k}\le p\le n.$$

Proof.

$$\begin{aligned}(f_k(x))^p &= (x+x^2+\cdots+x^k)^p \\ &= x^p(1+x+x^2+\cdots+x^{k-1})^p \\ &= x^p\left(\frac{1-x^k}{1-x}\right)^p \\ &= x^p\left((1-x^k)\left(\frac{1}{1-x}\right)\right)^p \\ &= x^p\left(\left(\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\right)\left(\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i\right)\right).\end{aligned}$$

In the above equation, we only consider the coefficient of x^n. Since the first term on the right is x^p, $kl+i=n-p$, that is, $i=n-kl-p$. If $l=q$ for any $q=0,1,\ldots,p$, then the second term on the right is

$$\left((-1)^q\binom{p}{q}\binom{n-kq-1}{n-kq-p}\right)x^{n-p}.$$

So, the coefficient of x^n is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \quad \frac{n}{k}\le p\le n.$$

The proof is completed.

Now we have a combinatorial representation for g_n.

Theorem 2.2. For positive integers p and n,

$$g_{n+1} = \sum_{\frac{n}{k} \le p \le n} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-kl-1}{n-kl-p}. \tag{2.2}$$

Proof. Since

$$G_k(x) = g_1 + g_2 x + g_3 x^2 + \cdots + g_{n+1} x^n + \cdots$$

$$= \frac{1}{1 - x - x^2 - \cdots - x^k},$$

the coefficient of x^n is the $n+1$st Fibonacci number, g_{n+1} , in G_k. And,

$$G_k(x) = \frac{1}{1 - x - x^2 - \cdots - x^k}$$

$$= \frac{1}{1 - f_k(x)}$$

$$= 1 + f_k(x) + (f_k(x))^2 + \cdots + (f_k(x))^n + \cdots \tag{2.3}$$

$$= 1 + f_k(x) + x^2 \sum_{l=0}^{2} \binom{2}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{i+1}{i} x^i +$$

$$\cdots + x^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i + \cdots .$$

Since we consider the coefficient of x^n, we only need the first $n+1$ terms on the right. The $(p+1)$st term in (2.3) is

$$x^p \sum_{l=0}^{p} \binom{p}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{p+i-1}{i} x^i.$$

So, $kl + i = n - p$, and $\frac{n}{k} \le p \le n$. Hence, by Lemma 2.1, we have (2.2).

If $k = 2$, then

$$G_2 = (1,\, 1,\, 2,\, 3,\, 5,\, 8,\, 13,\, 21, \dots)$$

is the Fibonacci sequence $\{F_n\}$. Since the generating function for $\{F_n\}$ is $G_2(x) = \frac{1}{1-x-x^2}$, and hence,

$$\begin{aligned} G_2(x) &= \frac{1}{1 - x(1+x)} \\ &= 1 + x(1+x) + x^2(1+x)^2 + \cdots + x^n(1+x)^n + \cdots . \end{aligned}$$

If the first $n+1$ terms on the right are examined in reverse order, it is seen that the coefficient of x^n in $G_2(x)$ is

$$1 + \binom{n-1}{1} + \binom{n-2}{2} + \cdots \tag{2.4}$$

as asserted. So, we have the following corollary.

Corollary 2.3. Let F_{n+1} be the $(n+1)$st Fibonacci number. Then

$$\begin{aligned} F_{n+1} &= \sum_{i=0} \binom{n-i}{i} \\ &= \sum_{\frac{n}{2} \le p \le n} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-2l-1}{n-2l-p}. \end{aligned}$$

Proof. By (2.2) and (2.4), the proof is completed.

3. Properties of k–Fibonacci Sequences. In this section, we give some properties for k–Fibonacci sequences. First, we have the following theorem by using vector recurrence relation (1.1).

Theorem 3.1 [3]. For positive integers n and m,

$$\begin{aligned} g_{n+m} = {} & g_n g_{m-(k-1)} + (g_n + g_{n-1}) g_{m-(k-2)} + \\ & (g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\ & + (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m. \end{aligned}$$

Proof. For G_k, $k \geq 2$, since $g_1 = g_2 = 1$, we can replace the matrix Q_k in (2.2) with

$$Q_k = \begin{bmatrix} 0 & g_1 & 0 & \cdots & 0 \\ 0 & 0 & g_1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & & g_1 \\ g_1 & g_1 & \cdots & g_1 & g_2 \end{bmatrix}.$$

Then

$$Q_k^n = \begin{bmatrix} g_{n-(k-1)} & g^{\dagger}_{1,2} & g^{\dagger}_{1,3} & \cdots & g^{\dagger}_{1,k-1} & g_{n-(k-2)} \\ g_{n-(k-2)} & g^{\dagger}_{2,2} & g^{\dagger}_{2,3} & \cdots & g^{\dagger}_{2,k-1} & g_{n-(k-3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ g_{n-1} & g^{\dagger}_{k-1,2} & g^{\dagger}_{k-1,3} & \cdots & g^{\dagger}_{k-1,k-1} & g_n \\ g_n & g^{\dagger}_{k,2} & g^{\dagger}_{k,3} & \cdots & g^{\dagger}_{k,k-1} & g_{n+1} \end{bmatrix},$$

where

$$\begin{aligned} g^{\dagger}_{i,2} &= g_{n-(k-i)} + g_{n-(k-(i-1))}, \\ g^{\dagger}_{i,3} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))}, \\ &\vdots \\ g^{\dagger}_{i,k-1} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} + \cdots + g_{n-(k-(i-(k-2)))}. \end{aligned}$$

Since $Q_k^n Q_k^m = Q_k^{n+m}$, $g_{n+m} = (Q_k^{n+m})_{k,1}$.

Therefore,

$$
\begin{aligned}
g_{n+m} &= g_n g_{m-(k-1)} + g^{\dagger}_{k,2} g_{m-(k-2)} + g^{\dagger}_{k,3} g_{m-(k-3)} + \cdots \\
&\quad + g^{\dagger}_{k,k-1} g_{m-1} + g_{n+1} g_m \\
&= g_n g_{m-(k-1)} + (g_n + g_{n-1}) g_{m-(k-2)} + (g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\
&\quad + (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m.
\end{aligned}
$$

We also have another representation of the nth k–Fibonacci number for positive intergers n and m.

<u>Corollary 3.2</u>. For positive integers n and m,

$$
\begin{aligned}
g_{n+m} = {} & g_{n-1} g_{m-(k-2)} + (g_{n-1} + g_{n-2}) g_{m-(k-3)} + \\
& (g_{n-1} + g_{n-2} + g_{n-3}) g_{m-(k-4)} + \cdots \\
& + (g_{n-1} + g_{n-2} + g_{n-3} + \cdots + g_{n-(k-1)}) g_m + g_n g_{m+1}.
\end{aligned}
$$

<u>Proof</u>. Since $g_{n+m} = (Q_k^{n+m})_{k,1} = (Q_k^{n+m})_{k-1,k}$, the proof is completed.

For example, for $n > k$,

$$
\begin{aligned}
g_{2n} &= g_{2n-1} + g_{2n-2} + \cdots + g_{2n-k} \\
&= g_{n+n} \\
&= g_{n-1} g_{n-(k-2)} + (g_{n-1} + g_{n-2}) g_{n-(k-3)} + \cdots \\
&\quad + (g_{n-1} + g_{n-2} + \cdots + g_{n-(k-1)}) g_n + g_n g_{n+1}.
\end{aligned}
$$

So, we can get g_{2n} by using g_{n+1}, $g_n, \dots,$ g_{n-k+2}.

The above fact raises a question [1, 2]. What is the relationship between g_n and g_{nt} for a positive integer t. In particular, is there a t such that g_n is a factor of g_{nt}? In the Fibonacci numbers, $F_n | F_{tn}$ for all $t = 1, 2, 3, \dots$. However, this is not true, in general, for k–Fibonacci numbers, $k \geq 3$.

<u>Lemma 3.3</u>. For any positive integer n, the k–Fibonacci numbers g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers.

Proof. If $n = 1$, then $g_1 = g_2 = 1$, i.e., g_1 and g_2 are odd numbers. By induction on n, we may assume true for n, and consider $n+1$.

First,

$$\begin{aligned} g_{(n+1)k+(n+1)-k} &= g_{nk+n+1} \\ &= g_{nk+n} + g_{nk+n-1} + \cdots + g_{nk+n-k+2} + g_{nk+n-k+1} \\ &= g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+1} + g_{nk+n-k} \\ &\quad + (g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+2} + g_{nk+n-k+1}) \\ &= 2(g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+1}) + g_{nk+n-k}. \end{aligned}$$

Then $g_{(n+1)k+(n+1)-k}$ is an odd number since g_{nk+n-k} is an odd number by hypothesis. Similarly, $g_{(n+1)k+(n+1)-k+1}$ is also an odd number.

Therefore, for any positive integer n, the k–Fibonacci numbers g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers.

Since $g_n^{(k)} = g_{n-k+2}$, our question can be replaced from "Is there any t such that $g_n^{(k)} | g_{nt}^{(k)}$ for some n?" to "Is there any t such that $g_{n-k+2} | g_{nt-k+2}$ for some n?"

Theorem 3.4. For $k \geq 3$, there exists t such that $g_{n-k+2} \not| g_{nt-k+2}$ for some n.

Proof. If $k = 3$, then

$$G_3 = (1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ 81,\ 147,\ \ldots).$$

Here, $g_4 = 4$, $g_9 = 81$ and hence, $g_4 \not| g_9$. In this case, $n = 5$ and $k = 3$.

Now, suppose that $k \geq 4$. Then, for any positive integer n, the g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers. Let $n = k+2$, $t = k$ and let $m = n+1$. Then

$$mt - k + 2 = (n+1)k - k + 2 = \left(n + \frac{n-2}{k}\right) k - k + 2 = nk + n - k.$$

So, g_{mt-k+2} is an odd number. Since $n = k+2$, $k \geq 4$ and $G_k = (1,\ 1,\ 2,\ 4,\ 8, \ldots)$,

$$g_{m-k+2} = g_{n+1-k+2} = g_5.$$

Since $k \geq 4$, in any cases, $g_5 = 8$. Since g_{m-k+2} is an even number and g_{mk-k+2} is an odd number, there exists t such that $g_{n-k+2} \not| g_{nt-k+2}$ for some n.

Now we have another question for any positive integers m and n. The question is "how many k–Fibonacci numbers are there between n^m and n^{m+1}?"

Lemma 3.5. For positive integers n and r,

$$ng_r \leq g_{r+n}. \tag{2.3}$$

Proof. If $n = 1$, then $g_r \leq g_{r+1}$. By induction on n, we may assume true for n, and consider $n + 1$. That is,

$$\begin{aligned} ng_r \leq g_{r+n} \;\Rightarrow\; & ng_r + g_r \leq g_{r+n} + g_r \\ \Rightarrow\; & (n+1)g_r \leq g_{r+n-1} + g_{r+n-2} + \cdots + g_{r+n-k} + g_r \\ & = g_{r+n} + (g_{r+n-1} + \cdots + g_{r+n-(k-1)} + g_{r+n-k} + g_r) - g_{r+n} \\ & = g_{r+n+1} + g_{r+n-k} + g_r - g_{r+n}. \end{aligned}$$

Since $g_{r+n} = g_{r+n-1} + g_{r+n-2} + \cdots + g_{r+n-k}$ and $n \geq 1$, $g_{r+n-k} + g_r \leq g_{r+n}$. Thus, $(n+1)g_r \leq g_{r+n+1}$.

Therefore, $ng_r \leq g_{r+n}$ for any positive integers n, r.

Theorem 3.6. Let m and n be any two positive integers. Then there are no more than n k–Fibonacci numbers between the consecutive powers n^m and n^{m+1}.

Proof. Suppose that the interval between some n^m and n^{m+1} were to contain at least $n + 1$ k–Fibonacci numbers:

$$n^m < g_{r+1},\ g_{r+2}, \ldots,\ g_{r+n+1}, \ldots < n^{m+1}.$$

Since $n^m < g_{r+1}$, $n \cdot n^m < ng_{r+1}$. So, by (2.3),

$$n^{m+1} < ng_{r+1} \leq g_{r+n+1}.$$

Consequently, $n^{m+1} < g_{r+n+1}$, a contradiction.

One of the most well–known properties of the Fibonacci sequence is the formula for the sum $S_n^{(2)}$ of the first n terms. A glance at the first few cases quickly leads to the conjecture

$$S_n^{(2)} = F_1 + F_2 + \cdots + F_n = F_{n+2} - 1,$$

which is immediately confirmed by mathematical induction. In case $k \geq 3$, we can easily verify that

$$S_n^{(k)} = \frac{1}{k-1}\left(g_{n+2}^{(k)} - g_{n+(k-2)}^{(k)} - 2g_{n+(k-3)}^{(k)} - \cdots - (k-2)g_{n+1}^{(k)} - 1\right).$$

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