## SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

**40**. [1991, 150; 1992, 150–151] *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

A tetrahedron is a geometric solid with 4 vertices, 6 edges, and 4 triangular faces. A Heron triangle is one whose sides and area are integers. A Heron tetrahedron is one having Heron triangles as faces and whose volume is an integer.

(a) Show that if $\triangle ABC$ is acute, then a tetrahedron exists with each of its faces congruent to $\triangle ABC$.

(b)* John Leech has shown that a Heron tetrahedron exists: Let $\triangle ABC$ have sides 148, 195, and 203 and let $T$ be the tetrahedron obtained from this triangle as in (a). Then each face of $T$ has integer area and $T$ has integer volume. The following question is inspired by Jim Buddenhagen's investigation of Heron triangles whose area is a square. Question: Is there a Heron tetrahedron whose volume is a perfect square or perfect cube?

*Comment by Les Reid, Southwest Missouri State University, Springfield, Missouri.* Once the existence of a Heron tetrahedron is known, it's relatively easy to construct a Heron tetrahedron whose volume is a perfect square (or, in fact, any perfect power whose exponent is not a multiple of 3). In general, if we scale the tetrahedron by a factor of $L$, the volume will increase by a factor of $L^3$ and the area by a factor of $L^2$ (so it will still be an integer). If we choose $L$ to be the square–free part of the volume, the volume will be a perfect square. For example, starting with Leech's tetrahedron having four congruent faces with edges of length 148, 195, and 203, it's volume is

$$611520 = 2^6 * 3 * 5 * 7^2 * 13,$$

whose square–free part is $3 * 5 * 13$. Therefore, the corresponding tetrahedron with edges of length 28860, 38025, and 39585 will have a volume of

$$2129400^2.$$

A similar argument works as long as the exponent of the power is not a multiple of 3. For example, if we want the volume to be a fifth power (and begin with Leech's tetrahedron), we would choose

$$L = 2^x * 3^y * 5^z * 7^s * 13^t,$$

so that $6+3x=0 \mod 5$, $1+3x=0 \mod 5$, $1+3z=0 \mod 5$, $2+3s=0 \mod 5$, and $1+3t=0 \mod 5$ (which have solutions since 3 is relatively prime to 5).

[I still haven't been able to find a volume that's a perfect cube.]

The question remains: "Are there primitive Heron tetrahedra whose volume is a perfect square?" [Here, primitive means the GCD of the edge lengths is 1.]

**100***. [1996, 136; 1997, 196–197] *Proposed by Bryan Dawson, Emporia State University, Emporia, Kansas.*

Let $C$ be the set of constructible numbers. Let $f\colon C \to \mathbb{R}$ be given by $f(x)=n$ where $n$ is the minimum number of arcs necessary to construct a segment of length $x$ under the following rules:

1) Only compass and straightedge may be used for the construction.
2) The construction starts with only a segment of unit length and this segment may not be used for any other purposes than measurement (i.e., the construction cannot be built using the segment; $f(1)=1$).
3) The number of uses of the straightedge must be finite.

Prove or disprove that $f$ has a point of continuity.

*Solution by Les Reid, Southwest Missouri State University, Springfield, Missouri.* With the author's stipulation that $f(1)=1$, the function $f\colon C \to \mathbb{R}$ is, in fact, constant (and hence, continuous everywhere). Once we draw one circle, any construction which can be done with compass and straightedge can be done with straightedge alone. This is an old result of Steiner [for example, see *100 Great Problems of Elementary Mathematics, Their History and Solution* by Heinrich Dorrie, pp. 165–170]. Therefore $f(x)=1$ for all $x$ in $C$.

**125**. [1999, 46] *Proposed by F. J. Flanigan, San Jose State University, San Jose, California.*

Let $d$ be a positive integer. The real polynomial $h(x)$ is $d$–simple on the open interval $(a,b)$ if and only if $h(x)=c(x-x_1)\cdots(x-x_e)$ with $0\le e\le d$ and the $x_i$ pairwise distinct elements of $(a,b)$. Let $f(x)$ be a real polynomial of degree $n\ge 1$. Prove that if

$$\int_a^b f(x)h(x)dx = 0$$

for all $d$–simple polynomials $h(x)$ on $(a,b)$, then

(i) $f(x)$ has at least $d+1$ roots of odd multiplicity in $(a,b)$, and
(ii) if $d=n-1$, then $f(x)$ has $n$ distinct simple roots in $(a,b)$.

*Solution by the proposer.* Note first that

$$\int_a^b f(x)\,dx = 0,$$

since $h(x) \equiv 1$ is $d$–simple, with degree $e = 0$ and $c = 1$. Thus, $f(x)$ must change sign at least once in $(a, b)$, and this implies that $f(x)$ has at least one root of odd multiplicity in $(a, b)$.

Next, suppose $f(x)$ has exactly $k \geq 1$ distinct roots of odd multiplicity in $(a, b)$. Thus,

$$f(x) = (x - x_1)^{m_1} \cdots (x - x_k)^{m_k} F(x)$$

with the $x_i$ pairwise distinct elements of $(a, b)$, the $m_i$ odd positive integers, and $F(x)$ a real polynomial which does not change sign in the interval $(a, b)$. Now consider the product

$$f(x)(x - x_1) \cdots (x - x_k) = (x - x_1)^{m_1+1} \cdots (x - x_k)^{m_k+1} F(x).$$

Observe that the polynomial on the right has no sign changes in $(a, b)$. Thus,

$$\int_a^b f(x)(x - x_1) \cdots (x - x_k) dx \neq 0.$$

Thus, $(x - x_1) \cdots (x - x_k)$ is not $d$–simple on $(a, b)$, which implies $k > d$. This proves (i).

If $d = n - 1$, it follows that $f(x)$ has $n$ distinct roots of odd multiplicity in $(a, b)$. But $f(x)$ has degree $n$, and it follows that each of these must have multiplicity $m_i = 1$, proving (ii).

**126**. [1999, 47] *Proposed by F. J. Flanigan, San Jose State University, San Jose, California.*

For which coefficient functions $b(t)$ will the equation

$$y''(t) + b(t)y'(t) + \sqrt{1 + y(t)^2} = 0$$

admit a solution $y(t)$ which is oscillatory on some interval, that is, rising and falling repeatedly (as, for example, a perturbed sine wave, or a polynomial with several real roots, etc.)?

*Solution by the proposer.* For no coefficient function $b(t)$ will this equation admit an oscillatory solution. This follows from the following proposition.

Proposition. If $y(t)$ is a solution to the given equation, then $y(t)$ has no local minimum.

Proof. If $t_0$ is a critical value for the solution $y(t)$, then $y'(t_0) = 0$, and so the differential equation implies

$$y''(t_0) = -\sqrt{1 + y(t_0)^2} \leq -1 < 0.$$

By the second derivative test of differential calculus, the critical value yields a strict local *maximum* for $y(t)$. This completes the proof.

It follows that a solution $y(t)$ cannot have two local maxima on an interval of definition, for between two local maxima there would be at least one local minimum. This paucity of local extrema rules out oscillatory behavior.

*Comments by the proposer.*

i) The above proof employs a rudimentary version of the 'maximum principle' (though here it would be the 'minimum principle,' with $y(t)$ attaining its minimum on a closed interval *only at an endpoint.*)

ii) One does not have to know 'how to solve differential equations' to work this problem.

iii) Note that if $b(t) \equiv 0$, then the equation of the problem is *roughly* similar to

$$y''(t) + y(t) = 0,$$

whose solutions *are* periodic and therefore oscillatory. But there is a crucial difference ... .

**127**. [1999, 47] *Proposed by Vincent Dunn (student) and Donald P. Skow, University of Texas – Pan American, Edinburg, Texas.*

If $x$ is a triangular number, $a$ and $b$ are positive integers, under what conditions is $ax + b$ also a triangular number? For example, $25x + 3$ satisifies the conditions.

*Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.* We shall prove the following theorem.

Theorem. If $x$ is any triangular number and $a$ and $b$ are positive integers, then $ax + b$ is a triangular number if and only if $a$ is an odd square and $b$ is the $k$th triangular number, where

$$k = \frac{\sqrt{a} - 1}{2}.$$

Proof. Throughout this proof, write $x$ in the form

$$\frac{n(n+1)}{2},$$

the general form of a triangular number.

Now assume $a = (2k+1)^2$, an odd square, so that

$$\frac{\sqrt{a}-1}{2} = \frac{2k+1-1}{2} = k,$$

and

$$b = \frac{k(k+1)}{2}$$

is the $k$th triangular number. This implies

$$\begin{aligned} ax+b &= \frac{(2k+1)^2 n(n+1)}{2} + \frac{k(k+1)}{2} \\ &= \frac{(2k+1)^2 n(n+1) + k(k+1)}{2} \\ &= \frac{((2k+1)n+k)((2k+1)n+(k+1))}{2} \\ &= \frac{m(m+1)}{2}, \end{aligned}$$

where $m = (2k+1)n + k$. Thus, $ax+b$ is a triangular number.

For the converse, now assume

$$ax+b = \frac{an(n+1)}{2} + b = \frac{an^2+an+2b}{2}$$

is a triangular number. If

$$\frac{an^2 + an + 2b}{2}$$

is to factor in the form

$$\frac{m(m+1)}{2},$$

then there must exist positive integers $c$ and $k$ such that

$$\begin{aligned} an^2 + an + 2b &= (cn + k)(cn + k + 1) \\ &= c^2 n^2 + (2ck + c)n + k^2 + k. \end{aligned}$$

This implies $a = c^2$, $a = 2ck + c$, and $2b = k^2 + k$. Then $c^2 = 2ck + c$, so that $c = 0$ or $c = 2k + 1$. $c = 0$ makes $a = 0$, not a positive integer, so disregard the solution $c = 0$.

Next consider the solution $c = 2k + 1$. This makes $c$ an odd integer, so that $a = c^2 = (2k + 1)^2$ is an odd square. Furthermore,

$$b = \frac{k(k+1)}{2}$$

is the $k$th triangular number, where

$$k = \frac{c-1}{2} = \frac{\sqrt{a} - 1}{2}.$$

This completes the proof.

*Also solved by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; N. J. Kuenzi and John Koker, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; David Lung (graduate student), University of Texas–Pan American, Edinburg, Texas; Kenneth B. Davenport; Frackville, PA; and the proposers.*

*Comment by James Bruening and the proposers.* David M. Burton, in his book *Elementary Number Theory,* Fourth Edition, The McGraw–Hill Companies, New York, 1998, p. 15, Problem 1(d), seems to credit Euler in 1775 with discovering these conditions. Burton, who is also a notable mathematics historian, lists many references in the back of his book (pp. 341–345) which could be checked to find a discussion of Euler's work with triangular numbers.

**128***. [1999, 47] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let $b \geq 2$ be a positive integer. Let $g(x)$ be periodic of period $1/(b-1)$ and on $[0, 1/(b-1)]$ be equal to the piecewise linear function connecting the points

$$\left(\frac{d}{b^2-b}, \frac{(b-d)d}{2b}\right),$$

where $d$ is a nonnegative integer and $0 \leq d \leq b$. Let

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{b^i} g(b^i x).$$

For $b = 2$ it is known that the maximum value of $f$ is $1/3$ and the set $E$ where this occurs is the set of values whose fractional part can be represented as the infinite quaternary fraction $0.\alpha_1\alpha_2\ldots\alpha_n\ldots$, where every $\alpha_i$ is either one or two.

What is the maximum value of $f$ for a general $b$ and what is the set $E$ where these maximum values occur?

*Comment by the editor.* No solution was received. Therefore, the problem remains open.