A NOTE ON THE ISOPERIMETRIC INEQUALITY

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In most calculus books students are introduced to an isoperimetric theorem in the following form. Among all rectangles with a given perimeter the square has the largest area [5]. However, this isoperimetric theorem for rectangles can be proved easily in an algebraic class using quadratic functions. The following theorem might be more appropriate for a calculus class. Among all quadrilaterals with a given perimeter and a given side, the trapezoid with the other sides of equal length, and of equal angles between them has the largest area. This isoperimetric theorem for quadrilaterals has a nice application, an isoperimetric theorem for n–polygons, i.e. polygons with n vertices. Among all n–polygons with a given side and a given perimeter, the n–polygon with the maximum area is inscribed in a circle and has all other sides of equal length and of equal angles between them where the existence of such an n–polygon is obtained from the general result that a continuous function on a bounded and closed subset of an Euclidean plane, $\mathbb{R}^N$, attains a maximum value.

In this note proofs of the above theorems are presented and the theorems are then utilized along with inscribed polygons to obtain a proof for the following isoperimetric theorem for simple closed curves. Let S be a closed curve formed by a circular arc of length s together with its chord of length ℓ. Then any simple closed curve Σ formed by a curve of length s together with a line segment of length ℓ satisfies the inequality $A(\Sigma) \leq A(S)$ where $A(\sigma)$ denotes the area enclosed by the simple closed curve σ and the equality holds if and only if Σ coincides with S. As a corollary we obtain the isoperimetric theorem for simple closed curves [2]. Any simple closed curve Σ with length s satisfies the inequality $4\pi A(\Sigma) \leq s^2$, with equality if and only if Σ is a circle.

In his paper [4] "The Isoperimetric Inequality," Professor Osserman obtains the following inequality for any n–polygon Σ with perimeter s.

$$\frac{s^2}{A(\Sigma)} \geq \frac{4}{n} \tan \frac{\pi}{n} > 4\pi.$$

He states that one can use inscribed polygons, apply the above inequality and pass to the limit to deduce $s^2 \geq 4\pi A(\sigma)$ for a simple closed curve σ with length s. "The shortcoming of this proof," he continues, "is that it does not allow a characterization of the case in which equality holds." In this paper we show that this shortcoming may be overcome by initially proving the isoperimetric inequality for simple closed curves with a fixed line segment. While we do not know whether the proof of the isoperimetric inequality is new, the methods used should be of interest to teachers and students in a first calculus course. All curves mentioned in this note are rectifiable plane curves.

We begin with an isoperimetric theorem for triangles which may be proved from the study of ellipses in an analytic geometry class. If P_1, P_2, P_3 are three different points in $\mathbb{R}^2$, then $\angle P_1P_2P_3$ denotes a number in $[-\pi, \pi]$ which measures the angle from the ray $\overrightarrow{P_2P_1}$ to the ray $\overrightarrow{P_2P_3}$ (a positive number if measured counterclockwise) and $A(\triangle P_1P_2P_3)$ denotes the area of the triangle $\triangle P_1P_2P_3$ if P_1, P_2, P_3 are non–colinear and 0 otherwise. The notation $|P_1P_2|$ denotes the length of P_1P_2, the line segment from P_1 to P_2.

Proposition 1. Among all triangles with a given side and a given perimeter the isosceles triangle has the largest area.

Proof. Instead of an algebraic proof, we present a proof where the concept of derivative is used. Any triangle $\triangle ABC$ (see Diagram 1) with a given side BC of length a and a given perimeter $a + s$ is determined by the angle $\angle CBA = \theta$ and has an area

$$f(\theta) = \frac{1}{2} ac \sin\theta$$

where $|AB| = c$, $|AC| = b$ and $c + b = s$. From the identity $(s - c)^2 = c^2 + a^2 - 2ac\cos\theta$, we get

$$c = \frac{s^2 - a^2}{2(s - a\cos\theta)}.$$

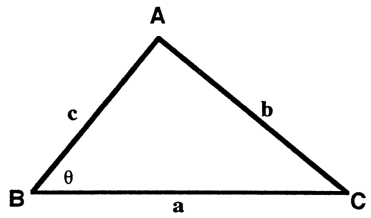

Diagram 1.

Substituting and differentiating we get

$$f'(\theta) = \frac{1}{4}a(s^2 - a^2)\frac{s\cos\theta - a}{(s - a\cos\theta)^2}.$$

Thus, f has the maximum value when $s\cos\theta = a$, i.e. when $\angle CBA = \angle ACB$.

Proposition 2. Among the quadrilaterals $ABCD$ with a given side AD and a given perimeter, the quadrilateral with the maximum area is the trapezoid satisfying the following (see Diagram 2):

(i) $|AB| = |BC| = |CD|$.
(ii) $\angle DAB = 2\angle CAB = \angle CDA = 2\angle DBC$ and $\angle ABC = \angle BCD$.
(iii) The quadrilateral $ABCD$ is symmetric with respect to the perpendicular bisector of AD.
(iv) There is a point O such that $|OA| = |OB| = |OC| = |OD|$.

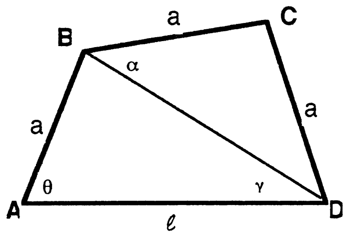

Diagram 2.

Proof. Since the quadrilateral with the largest area is convex, (i) follows from Proposition 1. Any convex quadrilateral $ABCD$ such that $|AD| = \ell$ and $|AB| = |BC| = |CD| = a$ with a and ℓ fixed is determined by the angle $\angle DAB = \theta$ (see Diagram 2) and has an area

$$f(\theta) = \frac{1}{2}(\ell a \sin\theta + 2a^2 \sin\alpha\cos\alpha) = \frac{a}{2}(\ell\sin\theta + a\sin 2\alpha)$$

where $\alpha = \angle DBC$. So $f'(\theta) = 0$ when

$$\frac{d\alpha}{d\theta} = -\frac{\ell\cos\theta}{2a\cos 2\alpha}.$$

Differentiating the identity $(2a\cos\alpha)^2 = a^2 + \ell^2 - 2a\ell\cos\theta$, we also get

$$\frac{d\alpha}{d\theta} = -\frac{\ell\sin\theta}{2a\sin 2\alpha}.$$

It follows that f has the maximum value when

$$\frac{\ell\cos\theta}{2a\cos 2\alpha} = \frac{\ell\sin\theta}{2a\sin 2\alpha}, \text{ i.e. } \tan\theta = \tan 2\alpha.$$

Consequently the quadrilateral $ABCD$ has the maximum area when $\theta = 2\alpha$.

Now suppose $\theta = 2\alpha$. Since $a\sin 2\alpha = 2a\cos\alpha\sin\gamma$ where $\gamma = \angle BDA$, we get

$$\sin\alpha = \sin\gamma \text{ and } \alpha = \gamma.$$

It follows that

$$AD \| BC \text{ and } \angle ABC = \angle BCD = \pi - 2\alpha.$$

The statements (ii) and (iii) now follow from equivalence of triangles. Finally the intersection point O of the bisectors of the angles $\angle ABC$ and $\angle BCD$ satisfies the conditions of (iv).

Now we will consider the case of n–polygons. First we show the existence of a polygon with the maximum area among the set of n–polygons with a given perimeter and a given side.

Proposition 3. Among all n–polygons with a given perimeter s and a given side of length ℓ there is a polygon with the maximum area.

Proof. Let Γ be the closed and bounded set of points $(a_{11}, \dots, a_{1n}, \dots, a_{n1}, \dots, a_{nn})$ in $\mathbb{R}^{n^2}$ satisfying the following conditions.

1. $A_1 = (0, \dots, 0)$, $A_n = (\ell, 0, \dots, 0)$,
2. $\sum_{i=1}^{n-1} |A_i A_{i+1}| + |A_n A_1| = s$, where $A_i = (a_{i1}, \dots, a_{in})$ for $i = 1, \dots, n$,
3. $0 \le \angle A_i A_1 A_{i-1}$ for $i = 3, \dots, n$, and $\sum_{i=3}^{n} \angle A_i A_1 A_{i-1} \le \pi$.

The function $f\colon \Gamma \to \mathbb{R}$ given by

$$f(a_{11}, \dots, a_{1n}, \dots, a_{n1}, \dots, a_{nn}) = \sum_{i=2}^{n-1} A(\triangle A_i A_1 A_{i+1})$$

is continuous, and so f attains the maximum value in Γ. Since a non–convex polygon cannot have the maximum area, the proof is complete.

Proposition 2, Proposition 3, mathematical induction and equivalence of triangles lead to the following theorem.

Theorem 4. Among the n–polygons with a given side and a given perimeter the n–polygon R_n which has the maximum area satisfies the following conditions.

(i) All sides with the possible exception of the given are of equal length.

(ii) All angles not touching the given side are of equal measure and so are the angles touching the given side.
(iii) The polygon R_n is symmetric with respect to the perpendicular bisector of the given side.
(iv) The polygon R_n is inscribed in a circle.

In particular, among all n–polygons with a given perimeter the regular n–polygon has the maximum area.

Proof. Let $R_n = A_1 A_2 \cdots A_{n-1} A_n A_1$ be the n–polygon with the given segment $A_1 A_n$ and with the maximum area (see Diagram 3). From Proposition 2, we conclude

$$|A_1A_2| = |A_2A_3| = \cdots = |A_{n-1}A_n| \text{ and}$$
$$\angle A_1A_2A_3 = \angle A_2A_3A_4 = \cdots = \angle A_{n-2}A_{n-1}A_n = \pi - 2\alpha$$

where $\alpha = \angle A_3A_1A_2$. From Proposition 2 and equivalence of triangles we get

$$\alpha = \angle A_{i+1}A_1A_i = \angle A_{j+1}A_nA_j \text{ for } i = 2, \dots, n-1, \quad j = 1, \dots, n-2.$$

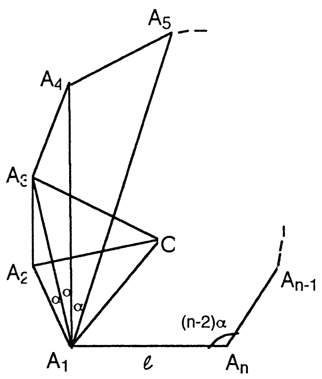

Diagram 3.

Therefore,

$$\angle A_nA_1A_2 = \angle A_{n-1}A_nA_1 = (n-2)\alpha$$

from which follow (ii) and (iii).

Let O be the intersection point of the bisectors of the angles $\angle A_1A_2A_3$ and $\angle A_2A_3A_4$. Then again from equivalence of triangles (iv) follows, i.e. all the vertices of the polygon R_n lie on the circle with center O and the radius equal to $|OA_1|$.

As an application of the theorem above, we present an isoperimetric theorem for simple closed curves containing a given segment and having a given length.

<u>Theorem 5</u>. Let Σ be a simple closed curve formed by a curve of length s together with a line segment of length ℓ $(s > \ell)$. Let $0 < \theta < \pi$ be defined by

$$\frac{\ell}{s} = \frac{\sin(\pi - \theta)}{\pi - \theta}.$$

Then

$$A(\Sigma) \leq A(S) = \frac{s}{4(\pi - \theta)}(s + \ell \cos\theta)$$

with equality if and only if Σ coincides with the simple closed curve S formed by a circular arc of length s and its chord of length ℓ.

<u>Remark</u>. The function

$$g(x) = \frac{\sin x}{x}$$

is strictly decreasing on the interval $(0, \pi)$ and thus θ in Theorem 5 is uniquely determined. See [1] for a simple proof which avoids the use of a formula of an area of a sector of a circle for

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which is used in the proof below.

<u>Proof</u>. Let PQ be the given segment. For each positive integer $n \geq 4$, let P_n be an n–polygon with perimeter $s_n + \ell$ with vertices in Σ which include P, Q and which divide Σ into n pieces of equal length. By Theorem 4, $A(P_n) \leq A(R_n)$ where $R_n = A_1A_2 \cdots A_nA_1$ is the n–polygons with perimeter $s_n + \ell$ such that

(i) R_n is inscribed in a circle with center O and radius r_n and
(ii) $|A_1A_n| = |PQ|$ and $|A_1A_2| = |A_2A_3| = \cdots = |A_{n-1}A_{n-2}|$.

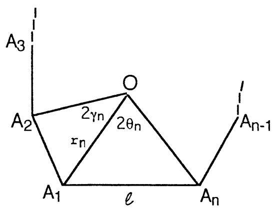

Diagram 4.

Let $\gamma_n = (1/2)\angle A_2OA_1$ and $\theta_n = \pi - (n-1)\gamma_n$ (see Diagram 4). Then since

$$r_n = \frac{\ell}{2\sin\theta_n} = \frac{s_n}{2(n-1)\sin\gamma_n}$$

we have

$$\frac{\ell}{s_n} = \frac{\sin\theta_n}{(n-1)\sin\left(\frac{\pi-\theta_n}{n-1}\right)}.$$

Taking limits as $n \to \infty$ we get

$$\frac{\ell}{s} = \lim_{n\to\infty} \frac{\sin(\pi-\theta_n)}{\pi-\theta_n}$$

and thus,

$$\lim_{n\to\infty} \theta_n = \theta.$$

The area $A(R_n)$ is given by

$$\begin{aligned} A(R_n) &= (n-1)r_n^2\cos\gamma_n\sin\gamma_n + r_n^2\cos\theta_n\sin\theta_n \\ &= \frac{r_n}{2}(s_n\cos\gamma_n + \ell\cos\theta_n) \\ &= \frac{r_n}{2}(s_n\cos\frac{\pi-\theta_n}{n-1} + \ell\cos\theta_n). \end{aligned}$$

Taking limits we arrive at the inequality.

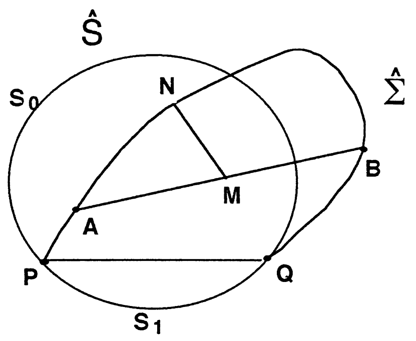

Diagram 5.

To prove the last assertion suppose Σ is convex and $A(\Sigma) = A(S)$. Let $S = S_0 \cup PQ$ where S_0 is the circular part of S with the end points P, Q. Let S_1 be the circular arc with the end points P, Q such that $\hat{S} = S_1 \cup S_0$ is a circle with circumference $\hat{s}$. Let $\hat{\Sigma} = (\Sigma - PQ) \cup S_1$ be the simple closed curve enclosing PQ (see Diagram 5). Since $A(\Sigma) = A(S)$ we have $A(\hat{\Sigma}) = A(\hat{S})$. It suffices to show that $\hat{\Sigma}$ and $\hat{S}$ coincide. Let A, B be points on $\hat{\Sigma}$ which divide $\hat{\Sigma}$ into two arcs of equal length. We may assume that AB divides the region enclosed by the curve $\hat{\Sigma}$ into two regions of equal area. We claim $|AB| = \hat{s}/\pi$. To prove this let Σ_1 be the simple closed curve which is the union of the circular arc of length $\hat{s}/2$ with the end points A, B and its reflection about AB. If $|AB| < \hat{s}/\pi$, the curve Σ_1 is not convex (see Diagram 6) and thus, by the first part of Theorem 5,

$$A(\hat{\Sigma}) \leq A(\Sigma_1) < A(\hat{S}) = \frac{\hat{s}^2}{4\pi},$$

a contradiction. If $|AB| = \hat{s}/\pi$ then the intersection points D, E of the bisector of AB with Σ_1 divide Σ_1 into two arcs of equal length (see Diagram 7) and $|DE| < \hat{s}/\pi$ and as before we get a contradiction.

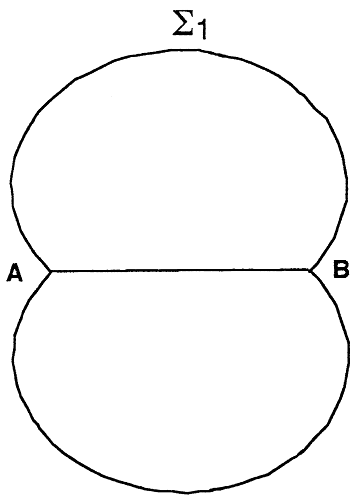

Diagram 6.

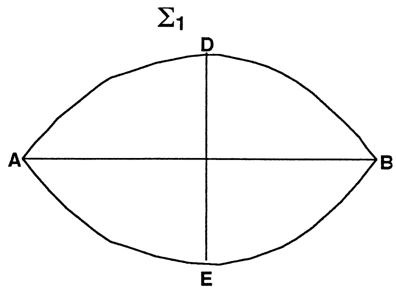

Diagram 7.

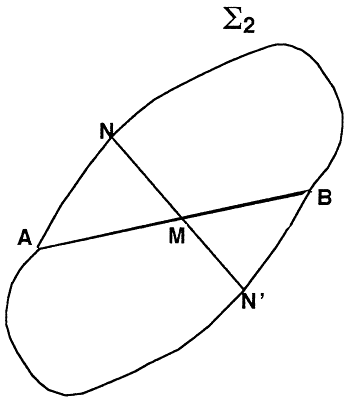

Diagram 8.

Finally suppose $|AB| = \hat{s}/\pi$ and $\hat{\Sigma}$ is not a circle. Let M be the midpoint of AB. There is a point N on $\hat{\Sigma}$ such that $|MN| < \hat{s}/2\pi$. Let Σ_2 be the union of the arc ANB of $\hat{\Sigma}$ and its reflection about M (see Diagram 8). Then N and its reflection point N' about M divide Σ_2 into two arcs of equal length and $|NN'| < \hat{s}/\pi$. As before then $A(\hat{\Sigma}) \leq A(\Sigma_2) < A(\hat{S})$, a contradiction and so $\hat{\Sigma}$ must be a circle and thus, $\Sigma = S$.

The following isoperimetric theorem for simple closed curves which has been given many proofs [2,4] follows from Theorem 5 with $\ell = 0$, i.e. when the given segment is a single point.

Corollary 6. If Σ is a simple closed curve with length s, then $4\pi A(\Sigma) \leq s^2$ with equality if and only if Σ is the circle with circumference s.

Among all simple closed curves which are formed by curves of length s together with line segments joining its end points, the one that bounds the maximum area must be formed by, from Theorem 5, a circular arc. It is known that among the circular arcs of a fixed length the smaller the curvature (i.e. the larger the diameter), the longer the chord [3]. The following corollary states that the diameter of the one which bounds the maximum area coincides with the length of its chord and can be proved by considering the closed curve formed with its mirror image.

Corollary 7. Let Σ be a simple closed curve formed by a circular arc of length s together with its chord. Then $2\pi A(\Sigma) \leq s^2$ with equality if and only if Σ is the curve formed by the semicircle of length s with its chord.

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