# NONZERO INJECTIVE COVERS OF MODULES

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**Abstract.** We show that if $R$ is a ring such that every nonzero left $R$–module has a nonzero injective cover, then $R$ is left Artinian. The converse is not true. If $R$ is commutative, then the properties are equivalent.

**1. Background.** A *projective cover* of a left $R$–module $M$ (first introduced by H. Bass [2] in the early 1960s) can be characterized as a pair $(P, f)$, where $P$ is a projective left $R$–module and $f\colon P \to M$ is a linear map satisfying the following two properties:

(i) if $P'$ is projective and $g\colon P' \to M$ is a linear map, then there exists a linear map $h\colon P' \to P$ such that $f \circ h = g$.
(ii) any linear map $h\colon P \to P$ such that $f \circ h = f$ is an automorphism of $P$.

The dual notion of an *injective envelope* of an $R$–module $M$ consists of an injective left $R$–module $E$ and a linear map $f\colon M \to E$ such that

(i) if $E'$ is injective and $g\colon M \to E'$ is a linear map, then there exists a linear map $h\colon E \to E'$ such that $h \circ f = g$, and
(ii) any linear map $h\colon E \to E$ such that $h \circ f = f$ is an automorphism of $E$.

Using these categorical definitions of projective cover and injective envelope as motivation, other types of covers and envelopes can be defined. This was initiated by E. Enochs in [4]. An *injective cover* of a left $R$–module $M$ consists of an injective module $E$ and a linear map $f\colon E \to M$ satisfying the following two properties:

(i) if $E'$ is injective and $g\colon E' \to M$ is a linear map, then there exists a linear map $h\colon E' \to E$ such that $f \circ h = g$.
(ii) any linear map $h\colon E \to E$ such that $f \circ h = f$ is an automorphism of $E$.

It was shown in [4] that a ring $R$ is left Noetherian if and only if every left $R$–module has an injective cover. However, we find that injective covers of many nonzero modules are zero. For example, the injective cover of $\mathbb{Z}$, or any finitely generated $\mathbb{Z}$–module, is zero. Therefore, we are interested in studying a nonzero map $\phi\colon E \to M$, where $M \neq 0$ is a left $R$–module, and $E$ is an injective module. Using the result cited above, it is easy to see that for a left Noetherian ring $R$, if every nonzero module $M$ has the property that there is a nonzero map $E \to M$ where $E$ is injective, then $M$ has a nonzero injective cover.

In this paper, $R$ will denote an associative ring with identity, and all modules will be unital left $R$–module. We denote by $l(x) = \{r \in R \mid rx = 0\}$ the left annihilator of $x \in R$, and if $R$ is commutative we use the notation $\mathrm{Ann}(x)$.

**2. Perfect Rings and Injective Modules.** Perfect rings were originally defined and studied using projective and flat modules. If $R$ is any ring, then by Theorem P of [2], $R$ is left perfect if and only if $R$ satisfies any one of the following equivalent conditions:

- every module has a projective cover.
- the Jacobson radical $J = J(R)$ is $T$–nilpotent and $R/J$ is Artinian
- every flat module is projective.

The results in this section provide a new way of viewing perfect rings by using injective modules.

Theorem 1. Let $J = J(R)$ be the Jacobson radical of a ring $R$ and suppose that for every $a \in R$, the set of left annihilator ideals $\{l(a^n) \mid n \geq 1\}$ satisfies the ascending chain condition (ACC). If for every nonzero left $R$–module $M$ there is a nonzero map $E \to M$ where $E$ is injective, then $J$ is a nil–ideal.

Proof. By Zorn's Lemma, there is a maximal nil–ideal $N$ contained in $J$. We claim that $N = J$. If $J/N \neq 0$, then there is a nonzero map $\phi\colon E \to J/N$ where $E$ is injective. Define $\mathrm{Tr}_{J/N}(E)$ to be the sum of all the images $h(E)$, where $h\colon E \to J/N$ is a linear map. That is, $\mathrm{Tr}_{J/N}(E) = \{\sum h(E) \mid h \in \mathrm{Hom}_R(E, J/N)\}$. Certainly $\mathrm{Tr}_{J/N}(E)$ is a submodule of $J/N$. Thus, $\mathrm{Tr}_{J/N}(E) = N_1/N$ for some ideal $N_1$ containing $N$, and $N_1/N \neq \overline{0}$.

For any $a \in N_1$, there is an integer $n$ such that $l(a^n) = l(a^{2n})$. Let $\overline{c} = \overline{a}^n = a^n + N$. Then $\overline{c} \in \mathrm{Tr}_{J/N}(E)$ and so there is an integer $t$ so that $\overline{c} = h_1(e_1) + \cdots + h_t(e_t)$, for some linear maps $h_i\colon E \to J/N$, and some elements $e_i \in E$, for $1 \leq i \leq t$. If we let $e^* = e_1 + \cdots + e_t$, let $E^* = E \oplus \cdots \oplus E$ denote $t$ copies of $E$, and let $h = \oplus_{i=1}^t h_i$, then $h\colon E^* \to J/N$, $e^* \in E^*$ and $h(e^*) = \overline{c}$. Define $f\colon Rc^2 \to E^*$ by $f(rc^2) = rce^*$. Then we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & Rc^2 \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\
& E^* &
\end{tikzcd}$$

and since $E^*$ is injective, there is a map $g$ which extends $f$. Let $e^* = g(1) \in E^*$. Then

$$ce^* = f(c^2) = g(c^2) = c^2 g(1) = c^2 e_1^*.$$

Now $h(ce^*) = h(c^2 e_1^*)$ implies that $\overline{c}^2(1 - h(e_1^*)) = \overline{0} \in J/N \subset R/N$. It follows that $\overline{c}^2 = \overline{0} \in R/N$, so $a^{2n} = c^2 \in N$ which means that $a$ is nilpotent in $R$. Therefore, $N_1$ is a nil–ideal of $R$, and $N \subset N_1$.

Corollary 1. If $R$ is left self–injective and for every $a \in R$ the set of left annihilator ideals $\{l(a^n) \mid n \geq 1\}$ has ACC, then $J(R)$ is nil and $R/J(R)$ is regular.

Proof. If $R$ is left self–injective, then every nonzero cyclic left $R$–module is a homomorphic image of an injective module, hence by Theorem 1, $J(R)$ is nil and by Anderson–Fuller, [1] $R/J(R)$ is regular.

Recall that a ring $R$ is said to be left perfect if and only if every left $R$–module has a projective cover. This is the case if and only if $J(R)$ is left T–nilpotent and $R/J(R)$ is Artinian semi–simple. See [2].

Theorem 2. If $R$ satisfies the following conditions, then $R$ is a left perfect ring.

(a) For every nonzero $R$–module $M$ there is a nonzero map $E \to M$ where $E$ is injective.

(b) For any sequence $\{a_n \mid n \geq 1\}$ in $R$, the set of left annihilator ideals $\{l(a_1 \cdots a_n) \mid n \geq 1\}$ has ACC.

(c) The union of any ascending chain of left T–nilpotent ideals is left T–nilpotent.

Proof. We will first show that the Jacobson radical $J$ is left T–nilpotent. By (c) and Zorn's Lemma, there is a maximal left T–nilpotent ideal $N \subset J$. If $J \neq N$, i.e. if $J/N \neq 0$, then by hypothesis (a) we have a nonzero map $\phi\colon E \to J/N$ where $E$ is injective. Set $\mathrm{Tr}_{J/N}(E) = L/N$, where $L$ is an ideal of $R$ containing $N$. By (b), for any sequence $\{a_n \in L \mid n \geq 1\}$, there is an integer $n_1$ such that $l(a_1 \cdots a_{n_1 - 1}) = l(a_1 \cdots a_{n_1})$. Now $\overline{a_{n_1}} = a_{n_1} + N \in L/N$. By definition of $\mathrm{Tr}_{J/N}(E)$, there is some integer $t$ so that $\overline{a_{n_1}} = h_1(e_1) + \cdots + h_t(e_t)$, where $h_i\colon E \to J/N$ is a linear map, and $e_i \in E$ for $i = 1, \ldots, t$. Let $E^*$ be the direct sum of $t$ copies of $E$, let $h = \oplus_{i=1}^t h_i$, and let $e^* = e_1 \oplus \cdots \oplus e_t$. Then $h\colon E^* \to J/N$ and $h(e^*) = \overline{a_{n_1}}$. Now define $f\colon Ra_1\cdots a_{n_1-1}a_{n_1}\to E^*$ by $f(ra_1\cdots a_{n_1-1}a_{n_1})=ra_1\cdots a_{n_1-1}e^*$, and let $g$ be a map which extends $f$. So we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Ra_1\cdots a_{n_1-1}a_{n_1} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\ & E^* & \end{tikzcd}$$

If we let $e_1^*=g(1)\in E^*$, then

$$\begin{aligned} &a_1\cdots a_{n_1-1}e^*=f(a_1\cdots a_{n_1})=a_1\cdots a_{n_1}e_1^*,\\ &h(a_1\cdots a_{n_1-1}e^*)=h(a_1\cdots a_{n_1}e_1^*),\\ &a_1\cdots a_{n-1}\overline{a_{n_1}}=a_1\cdots a_{n_1}h(e_1^*),\\ &\overline{a_1\cdots a_{n_1}}=\overline{a_1\cdots a_{n_1}}h(e_1^*). \end{aligned}$$

So $\overline{a_1\cdots a_{n_1}}(\overline{1}-h(e_1^*))=\overline{0}$ implies that $a_1\cdots a_{n_1}\in N$. Set $b_1=a_1\cdots a_{n_1}\in N$.

Similarly, for the sequence $\{a_k\mid k\geq n_1+1\}$, we can find an integer $n_2$ such that $a_{n_1+1}\cdots a_{n_2}\in N$. Set $b_2=a_{n_1+1}\cdots a_{n_2}$. Consequently we get a sequence $\{b_t\mid t\geq 1\}\subset N$, where $b_t=a_{n_t+1}\cdots a_{n_{t+1}}$ with $n_0=0$. Since $N$ is left T–nilpotent, for some integer $s$ we have $b_1b_2\cdots b_s=0$, and then $a_1\cdots a_{n_{s+1}}=0$. Hence $L$ is left T–nilpotent. It follows that $L=N$ is left T–nilpotent, and $\mathrm{Tr}_{J/N}(E)=L/N$ is zero. Therefore, $J=N$ is left T–nilpotent.

Next we will prove that $R/J$ is Artinian semi–simple. First of all, any set of orthogonal idempotent elements of $R/J$ can be lifted to a set of orthogonal idempotent elements of $R$. The set of left annihilator ideals $\{l(a_1\cdots a_n)\mid n\geq 1\}$ has ACC by assumption, and $R/J$ has an indecomposable decomposition as a left $R$–module. Let $\overline{R}=R/J=N_1\oplus\cdots\oplus N_t$, where each $N_i$ is an indecomposable submodule of $R/J$. Now we will show that $N_i$ is a simple $R/J$–module.

Let $H$ be a nonzero submodule of $N_i$. Since $N_i$ is indecomposable, in order to prove $H=N_i$ we only need to show that $H$ contains a nonzero direct summand of $N_i$. By the correspondence between idempotents and direct summands, we only need to construct a special idempotent element $w$ in $R/J$ such that it creates a direct summand of $N_i$.

By (a), there is an injective module $E$ and a map $\phi: E \to H$ with $\phi \neq 0$. Set $\mathrm{Tr}_H(E) = I/J \subset H$, and $I/J \neq 0$. For any $a \in I$, by (b) there is an integer $n$ such that $l(a^n) = l(a^{n+2})$. Now $\overline{a} = a + J \in \mathrm{Tr}_H(E)$, so there is an integer $t$ such that $\overline{a} = h_1(e_1) \oplus \cdots \oplus h_t(e_t)$ for some linear maps $h_i: E \to H$ and for some $e_i \in E$. If we let $E^*$ be the direct sum of $t$ copies of $E$, let $e^* = e_1 \oplus \cdots \oplus e_t$ and let $h = \oplus_{i=1}^t h_i$, then $h: E^* \to H$ and $h(e^*) = \overline{a}$. Now define $f: Ra^{n+2} \to E^*$ by $f(ra^{n+2}) = ra^n e^*$, and let $g$ be a map which extends $f$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Ra^{n+2} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\ & E^* & \end{tikzcd}$$

and if we set $g(1) = e_1^* \in E^*$, then

$$a^n e^* = f(a^{n+2}) = g(a^{n+2}) = a^{n+2} e_1^*.$$

Since $h(E^*)$ is contained in $\mathrm{Tr}_H(E) = I/J$, if we set $\overline{b} = h(e_1^*) \in I/J$, then since $h(e^*) = \overline{a}$,

$$\overline{a}^{n+1} = \overline{a}^{n+2}\overline{b}.$$

We can also find an integer $m$ and an element $\overline{c} \in I/J$ such that $\overline{b}^m = \overline{b}^{m+1}\overline{c}$. Consequently,

$$\overline{a}^{(n+1)+m+(n+1)+1}\overline{b}^{m+(n+1)+1} = \overline{a}^{n+1},$$

$$\overline{a}^{2(m+(n+1)+1)}\overline{b}^{m+(n+1)+1} = \overline{a}^{(n+1)+m+1},$$

$$\overline{b}^{m+(m+n+1)+1}\overline{c}^{m+n+1+1} = \overline{b}^m,$$

$$\overline{b}^{2(m+(n+1)+1)}\overline{c}^{m+(n+1)+1} = \overline{b}^{m+(n+1)+1}.$$

Set $x = \overline{a}^{n+1+m+1}$, $y = \overline{b}^{m+n+1+1}$ and $z = \overline{c}^{m+n+1+1}$. Then $x^2y = x$ and $y^2z = y$. And because $z - x \in I/J$, there is an integer $k$ and an element $d \in I/J$ such that $(z-x)^{k+1}d = (z-x)^k$. From these three relations

$$\begin{aligned} &x^2y = x, \\ &y^2z = y, \\ &(z-x)^{k+1}d = (z-x)^k, \end{aligned}$$

we can prove that $x = xyx$. (See the Appendix.)

Since $x \neq 0$, if we set $w = yx$, then $w \in I/J \subset H$ and $w^2 = yxyx = yx = w$. Hence, $\overline{R} = \overline{R}w \oplus K$, where $\overline{R}w \subset N_i$, and $N_i = \overline{R}w \oplus (K \cap N_i)$. This implies that $N_i = H$, since $N_i$ is indecomposable, and then $R/J$ is Artinian semi–simple.

Corollary 2. If every nonzero left $R$–module has a nonzero injective cover, then $R$ is left Artinian.

Proof. If every left $R$–module has an injective cover, then $R$ is left Noetherian, and it follows that the hypotheses (a), (b) and (c) in Theorem 2 are all satisfied. Hence, $R$ is left Artinian.

The following example shows that the converses to Theorem 2 and Corollary 2 are not true. We will give an example of a left Artinian ring $R$ for which hypothesis (a) of Theorem 2 is not satisfied.

Let $F$ be a field, let

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and let

$$R = Fe_{11} + Fe_{12} + Fe_{22} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in F \right\}.$$

We know that $R$ is left Artinian. Now we will show that the hypothesis (a) is not satisfied by $R$.

Since $R = Re_{11} \oplus (Re_{12} + Re_{22})$, $Re_{11}$ is a projective simple module of $R$. If there is a nonzero map $E \to Re_{11}$ with $E$ injective, then $Re_{11}$ is injective.

However, the homomorphism $f\colon Re_{12} \to Re_{11}$ given by $f(re_{12}) = re_{11}$, can not be extended. For if we consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & Re_{12} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\ & Re_{11} & \end{tikzcd}$$

where $g \mid Re_{12} = f$, if

$$g(1) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in Re_{11},$$

then $f(e_{12}) \neq 0$ but

$$g(e_{12}) = e_{12}g(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 3. If $R$ is commutative, then the following statements are equivalent.

1. $R$ is perfect.

2. (a) For every nonzero $R$-module $M$ there is a nonzero linear map $E \to M$ where $E$ is injective.

(b) For any sequence $a_1, a_2, \ldots$ in $R$, the set of annihilator ideals $\{\mathrm{Ann}(a_1 \ldots a_n) \mid n \geq 1\}$ has ACC.

Proof. (1) $\Longrightarrow$ (2). Because $R$ has DCC on principal ideals [2], there is an integer $n$ such that $Ra_1 \cdots a_n = Ra_1 \cdots a_{n+k}$, for all $k \geq 1$. This implies that $\mathrm{Ann}(a_1 \cdots a_n) = \mathrm{Ann}(a_1 \cdots a_{n+k})$ for $k \geq 1$. Hence, (b) is satisfied. For (a), consider the decomposition $R = R_1 \oplus \cdots \oplus R_t$, where each $R_i$ is a local ring with maximal ideal $m_i$. For any nonzero $R_i$–module $M_i$, the simple module $S_i = R_i/m_i$ can be embedded in $M_i$. Let $E_i(M_i)$ be the injective envelope of $M_i$ as an $R_i$–module. Then we have a nonzero map $E_i(M_i) \to E_i(M_i)/m_iE_i(M_i) \to S_i \hookrightarrow M_i$.

Since $R$ is commutative, $E_i(M_i)$ is $R$–injective. Therefore, for any nonzero $R$–module $M$, we have $M = R_1M \oplus \cdots \oplus R_tM$, where each $M_i = R_iM$ is an $R_i$–module. Now it is easy to see that there is a nonzero linear map $E \to M$ where $E$ is injective.

(2) $\Longrightarrow$ (1). By Theorem 1, $J$ is nil. For any sequence $\{a_n \mid n \geq 1\}$ in $J$, there is an integer $n$ such that $\mathrm{Ann}(a_1 \cdots a_{n+1}) = \mathrm{Ann}(a_1 \cdots a_n)$. Since $a_{n+1}$ is nilpotent, $a_{n+1}^k = 0$ and $a_{n+1}^{k-1}a_1 \cdots a_n a_{n+1} = 0$. Then $a_{n+1}^{k-1} \in \mathrm{Ann}(a_1 \cdots a_n)$, and so $a_{n+1}^{k-1}a_1 \cdots a_n = 0$. Continuing in this way we eventually get that $a_{n+1}^2 a_1 \cdots a_n = 0$, which implies that $a_{n+1} \in \mathrm{Ann}(a_1 \cdots a_n)$. This shows that $J$ is T–nilpotent. Then by the proof of Theorem 2, $R$ is perfect.

Corollary 3. If $R$ is commutative, then $R$ is Artinian if and only if every nonzero $R$–module has a nonzero injective cover.

**3. Appendix.** The purpose of this appendix is to prove the following proposition. The main idea of the proof is contained in the article by M. F. Dischinger [3].

Proposition 1. If $x$, $y$, $z$, $d$ are elements of $R$, and if

(i) $x^2y = x$,

(ii) $y^2z = y$,

(iii) $(z-x)^{k+1}d = (z-x)^k$ for some positive integer $k$,

then $xyx = x$.

Proof. Using (i) and (ii) it is easy to see that

$$xz = x^2, \tag{1}$$

$$xyz = x. \tag{2}$$

The first equation follows because $xz = x^2yz = x(x^2y)yz = xx^2y = xx$. The second equation follows by a similar calculation: $xyz = x^2yyz = x^2y = x$.

Using equation (1), we have $(z-x)^2 = z^2 - zx - xz + x^2 = z^2 - zx$ and so

$$(z-x)^2 = z(z-x).$$

Then an induction argument shows that for any integer $n \geq 1$,

$$(z-x)^n = z^{n-1}(z-x). \tag{3}$$

Now use this and equations (1) and (2) to see that

$$xy(z-x)^2 = xyz(z-x) = x(z-x) = xz - x^2 = 0. \tag{4}$$

Also, by hypothesis (iii), there is some positive integer $k$ so that

$$(z-x)^k = (z-x)^{k+1}d = (z-x)^k(z-x)d = z^{k-1}(z-x)^2d.$$

Using hypothesis (ii) as the basis for another induction argument, it follows that $y^n z^{n-1} = y$ for every positive integer $n$. Hence by (3),

$$y^n(z-x)^n = y^n z^{n-1}(z-x) = y(z-x) \tag{5}$$

for every positive integer $n$. Now using equation (3), hypothesis (iii) and the equation above, we have

$$y(z-x)^2d = y^k z^{k-1}(z-x)^2d = y^k(z-x)^{k+1}d = y^k(z-x)^k = y(z-x) \tag{6}$$

for some positive integer $k$. By equation (5) and this equation,

$$y(z-x)d = y^2(z-x)^2d = y^2(z-x).$$

By equations (4) and (6)

$$xy(z-x) = xy(z-x)^2d = 0.$$

Therefore, $xyz = xyx$. Now since $x = xyz$, we conclude that $xyx = x$.

## *References*


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