DEMOIVRE'S FORMULA TO THE RESCUE

Bella Wiener and Joseph Wiener

1. Introduction. Euler's formulas

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{1}$$

and DeMoivre's theorem

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx \tag{2}$$

make an inseparable and elegant team. A few months ago, we spoke on this theme in San Antonio [1] and it was a real treat to read papers [2] and [3], indeed. Article [3] uses Euler's formulas to express the integer powers of sine and cosine as trigonometric polynomials (finite linear combinations of sines and cosines of multiple angles) and thus, to calculate important trigonometric integrals. In fact, a lot of earlier works, including [4] and [5], accomplished the same act and, we trust, the method has fascinated us since the times of Euler and Laplace. Then, why don't our calculus texts accept this technique of renowned experts? The reason, in our opinion, is quite clear: Euler's formulas look very nice and sleek, but their theory is both complex and deep.

In this connection, let us quote what the *Monthly* editorial [6] once wrote: "Judging from the volume of mail addressed to *Classroom Notes* it appears that one of the greatest mysteries of undergraduate mathematics is the equation of Euler $e^{ix} = \cos x + i\sin x$. The objective of these correspondents is to develop this formula without the use of infinite series; and judging from the desperate devices employed by these writers in seeking to attain this end, it is highly desirable that a rigorous simple proof of this formula be available .... The first point to be emphasized is that the expression e^{ix} has to be *defined*, and that certain properties must be ascribed to it. Otherwise any proof falls to the ground. Rigorous treatments of this appear in the classical literature; for example, see G. H. Hardy, *Pure Mathematics*, p. 409 (fifth edition), or E. T. Whittaker and G. N. Watson, *Modern Analysis*, p. 581 (fourth edition). Since these have more general objectives in view it may be complained that they are too complicated for the purpose of defining the relatively simple expression e^{ix}. If, on the other hand, one wishes simplicity, he may directly define e^{ix} to be the expression $\cos x + i \sin x$. This would settle the whole matter, but such a definition is unsatisfactory on intuitive grounds and appears to be drawn out of the air. It is hoped that the following definition is satisfactory on all three grounds: rigor, simplicity, and intuition: e^{ix} is a complex valued function of the real variable x having the properties $e^{i0} = 1$; $de^{ix}/dx = ie^{ix}$." A theorem is then proved that

$$e^{ix} = \cos x + i \sin x.$$

This report illuminates the longstanding dilemma on the place of Euler's formulas in elementary calculus: they are tempting to use and impossible to prove. In the calculus books, they are derived by formally replacing x with the complex variable ix $\left(i = \sqrt{-1}\right)$ in the power series of the exponential function and comparing it with the series for $\cos x$ and $\sin x$. But wait, for the integration techniques it's a bit too late.

The purpose of this paper is to extend the ideas and results of note [7] and to show that DeMoivre's theorem (which is part of a standard trigonometry course) can do more than Euler's formulas, not less, with a higher degree of success. The proof of the theorem is simple: by cosine and sine of the sum formulas, the product of two complex numbers

$$z_1 = r_1 (\cos\theta_1 + i \sin\theta_1) \quad \text{and} \quad z_2 = r_2 (\cos\theta_2 + i \sin\theta_2)$$

is found to be

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right],$$

and this result extends to any number of terms. In the case when each of the n factors equals

$$z = r (\cos\theta + i \sin\theta),$$

their product becomes

$$z^n = r^n (\cos n\theta + i \sin n\theta), \tag{3}$$

which is the well–known DeMoivre formula.

2. Integrating Powers of Sine and Cosine. Integrals of even powers of sine and cosine are notoriously difficult, and most texts approach them either by the half-angle identities for $\cos^2 x$ and $\sin^2 x$ or via reduction formulas. DeMoivre's theorem allows closed formulas for those integrals to be derived quite easily. Let

$$z = \cos x + i \sin x; \tag{4}$$

then

$$\frac{1}{z} = \cos x - i \sin x, \tag{5}$$

and therefore,

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right). \tag{6}$$

Applying the binomial formula to (6) gives

$$\cos^m x = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} z^{m-k}\left(z^{-1}\right)^k = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} z^{m-2k}$$

and

$$\sin^m x = \frac{1}{(2i)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k z^{m-2k}.$$

From DeMoivre's formula (2) it follows that

$$z^{m-2k} = \cos(m-2k)x + i\sin(m-2k)x,$$

which transforms the latter equations into

$$\cos^m x = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\cos(m-2k)x + \frac{i}{2^m}\sum_{k=0}^{m}\binom{m}{k}\sin(m-2k)x \tag{7}$$

and

$$\sin^m x = \frac{1}{(2i)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\cos(m-2k)x$$

$$+ \frac{i}{(2i)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\sin(m-2k)x. \tag{8}$$

It remains to consider the following two cases.

Case 1: $m = 2n$ is even.
Note that the left–hand part of (7) is real and therefore,

$$\cos^{2n} x = \frac{1}{4^n} \sum_{k=0}^{2n} \binom{2n}{k} \cos{(2n-2k)}\, x.$$

Since

$$\binom{2n}{k} \cos{(2n-2k)}\, x + \binom{2n}{2n-k} \cos{(2k-2n)}\, x = 2\binom{2n}{k} \cos{(2n-2k)}\, x$$

and $\cos{(2n-2n)}\, x = 1$, then

$$\cos^{2n} x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k} \cos{(2n-2k)}\, x \right].$$

Finally, the substitution $j = n - k$ changes this formula to

$$\cos^{2n} x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{j=1}^{n} \binom{2n}{n-j} \cos 2jx \right]. \tag{9}$$

Similarly, since $i^{2n} = (-1)^n$, equation (8) leads to the expansion

$$\sin^{2n} x = \frac{1}{4^n} \left[\binom{2n}{n} + 2 \sum_{j=1}^{n} \binom{2n}{k} (-1)^j \cos 2jx \right]. \tag{10}$$

From here,

$$\int \cos^{2n} x\, dx = \frac{1}{4^n}\left[\binom{2n}{n} x + \sum_{j=1}^{n} \frac{1}{j}\binom{2n}{n-j}\sin 2jx\right] + C \tag{11}$$

and

$$\int \sin^{2n} x\, dx = \frac{1}{4^n}\left[\binom{2n}{n} x + \sum_{j=1}^{n} \frac{(-1)^j}{j}\binom{2n}{n-j}\sin 2jx\right] + C. \tag{12}$$

Case 2: $m = 2n+1$ is odd.
Applying to (7) the previous technique yields at once the result

$$\cos^{2n+1} x = \frac{1}{4^n}\sum_{j=0}^{n}\binom{2n+1}{n-j}\cos(2j+1)\,x. \tag{13}$$

On the other hand, for odd m, the first term on the right of (8) is imaginary and the second is real. Hence,

$$\sin^{2n+1} x = \frac{1}{2\cdot 4^n}\sum_{k=0}^{2n+1}\binom{2n+1}{k}(-1)^{n-k}\sin(2n+1-2k)\,x.$$

Again, we collect the equal terms corresponding to indices k and $2n+1-k$ and obtain the formula

$$\sin^{2n+1} x = \frac{1}{4^n}\sum_{j=0}^{n}\binom{2n+1}{n-j}(-1)^j\sin(2j+1)\,x. \tag{14}$$

Equations (13) and (14) produce the integrals

$$\int \cos^{2n+1} x\, dx = \frac{1}{4^n}\sum_{j=0}^{n}\frac{1}{2j+1}\binom{2n+1}{n-j}\sin(2j+1)\,x + C \tag{15}$$

and

$$\int \sin^{2n+1} x\,dx = \frac{1}{4^n}\sum_{j=0}^{n}\frac{(-1)^{j+1}}{2j+1}\binom{2n+1}{n-j}\cos(2j+1)\,x + C. \tag{16}$$

3. Examples and Remarks. Wallis's formulas

$$\int_0^{\pi/2}\cos^{2n} x\,dx = \int_0^{\pi/2}\sin^{2n} x\,dx = \frac{(2n)!}{4^n\,(n!)^2}\cdot\frac{\pi}{2} \tag{17}$$

follow at once from (11) and (12) by taking the integrals between 0 and $\pi/2$. In [8], formulas (17) were obtained by differentiating the integral

$$\int_0^{\infty}\frac{dy}{y^2+p} = \frac{\pi}{2}p^{-1/2}, \quad p > 0$$

n times with respect to p and letting $p = a^2$, $y = a\tan x$. Conversely, the difficult indefinite integral of $\left(y^2+1\right)^{-n-1}$ is reduced by the substitution $y = \tan x$ to the integral of $\cos^{2n} x$. The antiderivatives of $\cos^{2k} x \sin^{2m} x$ can be transformed to forms (11) or (12) by means of the identity $\sin^2 x + \cos^2 x = 1$. For the integrals of $\cos^{2k+1} x \sin^{2m} x\,dx$ and $\cos^{2k} x \sin^{2m+1} x\,dx$, the substitutions $u = \sin x$ and $u = \cos x$, respectively, are simpler and more efficient than DeMoivre's or Euler's formulas. DeMoivre's theorem can also be used to find closed expressions for certain sums involving the binomial coefficients.

Example. Writing the complex number $1+i$ in trigonometric form

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

and applying DeMoivre's formula gives

$$(1+i)^n = 2^{n/2}\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right).$$

On the other hand, by the binomial development,

$$(1+i)^n = 1 + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i + \binom{n}{4} + \binom{n}{5}i - \cdots,$$

and equating the corresponding real and imaginary parts in the latter equations yields the identities

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2}\cos\frac{\pi n}{4} \tag{18}$$

and

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{n/2}\sin\frac{\pi n}{4}. \tag{19}$$

Furthermore, adding and subtracting the expansions

$$2^n = (1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots$$

and

$$0 = (1-1)^n = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots$$

shows that

$$1 + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots = 2^{n-1} \tag{20}$$

and

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \cdots = 2^{n-1}. \tag{21}$$

Combining (18) with (20) and (19) with (21) generates not so trivial combinatorial identities

$$1 + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \cdots = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\cos\frac{\pi n}{4}\right) \tag{22}$$

and

$$\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \binom{n}{13} + \cdots = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\sin\frac{\pi n}{4}\right). \tag{23}$$

In conclusion, we want to emphasize once more that formula (6) does not assume that $z = e^{ix}$, and their derivation uses solely the idea that the reciprocal of the complex number $z = \cos x + i \sin x$ is equal to its conjugate.

References

1. B. Wiener and J. Wiener, "DeMoivre's Theorem in Summation and Integration Formulas," *Abstracts of Papers Presented to the American Mathematical Society*, 20 (1999), 150.

2. E. Cho, "Euler's Formula and DeMoivre's Formula for Quaternions," *Missouri Journal of Mathematical Sciences*, 11 (1999), 80–83.

3. T. Dana–Picard and D. Cohen, "Linearization of Trigonometric Polynomials and Integrals," *Missouri Journal of Mathematical Sciences*, 11 (1999), 87–92.

4. J. H. Mathews, "A Note on Integrating $\cos^{2n} x$ and $\sin^{2n} x$," *Mathematics and Computer Education*, 24 (1990), 34–36.

5. P. Glaister, "A Method for Determining Some Integrals," *Mathematics and Computer Education*, 25 (1991), 31–32.

6. C. B. Allendoerfer, "The Proof of Euler's Equation," *American Mathematical Monthly*, 55 (1948), 94–95.

7. J. Wiener, "Integrals of $\cos^{2n} x$ and $\sin^{2n} x$," *College Mathematics Journal*, 31 (2000), 60–61.

8. J. Wiener, "An Analytical Approach to a Trigonometric Integral," *Missouri Journal of Mathematical Sciences*, 2 (1990), 75–77.

Bella Wiener
Department of Mathematics
University of Texas – Pan American
Edinburg, TX 78539
email: bwiener@panam.edu

Joseph Wiener
Department of Mathematics
University of Texas – Pan American
Edinburg, TX 78539
email: jwiener@panam.edu