SOLUTION OF A CLASS OF COMPLEX VECTOR LINEAR FUNCTIONAL EQUATIONS

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Abstract. In this paper a class of complex vector linear functional equations is solved in the general case, so that the results obtained generalize the results given in [1].

Introduction. First we introduce the following notations.

Let $\mathcal{V}$ be a finite dimensional complex vector space and let there exist mappings $f_i\colon \mathcal{V}^p \to \mathcal{V}$ $(1 \leq i \leq n)$. Throughout this paper $\mathbf{Z}_i$ $(1 \leq i \leq n)$ are vectors in $\mathcal{V}$ and C_i are constant vectors in the same space. We may assume that $\mathbf{Z}_i = (z_{i1}(t), \dots, z_{in}(t))^T$, where the components $z_{ij}(t)$ $(1 \leq i \leq p;\ 1 \leq j \leq n)$ are complex functions and $\mathbf{O} = (0, 0, \dots, 0)^T$ is the zero vector in $\mathcal{V}$.

In the present paper we will solve the following complex vector functional equation

$$\sum_{i=1}^{n} f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (\mathbf{Z}_{n+i} \equiv \mathbf{Z}_i). \tag{1}$$

A special case of the above functional equation if $p < n < 2p-1$ is solved in [1]. Some other particular cases of the functional equation (1) are considered in [2,3] under the hypothesis that the functions and the independent variables are real.

If we put $f_i = a_i f$ $(1 \leq i \leq n)$, where a_i are complex constants, into (1), then we obtain the following functional equation

$$\sum_{i=1}^{n} a_i f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (\mathbf{Z}_{n+i} \equiv \mathbf{Z}_i) \tag{2}$$

which is a special case of (1). Functional equation (2) will be solved here.

1. Main Results.

We will prove the following results.

Theorem 1. The general solution of complex vector functional equation (1) is given by the following formulas.

$$f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) \tag{3}$$

$$= \sum_{j=1}^{i-1} (-1)^{i+1} F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{j=i+1}^{n} (-1)^{j} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \quad (1 \le i \le n),$$

where F_{ij} $(1 \le i \le n-1,\ i+1 \le j \le n)$ are arbitrary complex vector functions from $\mathcal{V}$ such that

$$F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) = A_{ij}$$

if

$$\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\} = \emptyset,$$

where A_{ij} are constant complex vectors from $\mathcal{V}$ and $\sum_a^s = \mathbf{O}$ $(a > s)$.

Proof. We will prove the statement of the theorem by mathematical induction. For $n = 2$, equation (1) becomes

$$f_1(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) + f_2(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{p+1}) = \mathbf{O}. \tag{4}$$

Putting $\mathbf{Z}_1 = C_1$ into equation (4), we get

$$f_2(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_{p+1}) = -F_{12}(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_p), \tag{5}$$

where the following notation is introduced

$$F_{12}(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_p) = f_1(C_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p).$$

If we put (5) into (4), we get

$$f_1(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = F_{12}(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_p). \tag{6}$$

Therefore, for $i = 1, 2$ from (3) we obtain (6) and (5), respectively, which means that the theorem holds for $n = 2$.

For some fixed n we suppose that the general solution of functional equation (1) is given by (3).

Now, consider the functional equation

$$\sum_{i=1}^{n+1} g_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) = \mathbf{O}. \tag{7}$$

If we put $\mathbf{Z}_i = C_i$ $(1 \leq i \leq n)$ into (7), we find that function g_{n+1} has the following form

$$\begin{aligned} &g_{n+1}(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{n+p}) \\ &= \sum_{j=1}^{n} (-1)^n F_{j,n+1}(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{n+p}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}). \end{aligned} \tag{8}$$

Substituting (8) into (7) and introducing the new notations

$$\begin{aligned} &f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) = g_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) \\ &+ (-1)^n F_{i,n+1}(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{n+p}\} \cap \{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\}) \quad (1 \leq i \leq n), \end{aligned} \tag{9}$$

we obtain equation (1). According to the inductive hypothesis, the general solution of this equation is given by the formulas in (3). Therefore, from (3), (8), and (9) we get

$$\begin{aligned} &g_i(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\}) \\ &= \sum_{j=1}^{i-1} (-1)^{i+1} F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \\ &+ \sum_{j=i+1}^{n} (-1)^j F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \\ &- (-1)^n F_{i,n+1}(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{n+p}\} \cap \{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\}) \quad (1 \leq i \leq n), \end{aligned}$$

i.e.

$$g_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1})$$

$$= \sum_{j=1}^{i-1} (-1)^{i+1} F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{j=i+1}^{n+1} (-1)^j F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$(1 \le i \le n+1).$$

The theorem holds for $n+1$ if it holds for n.

Example 1. The general solution of the complex vector functional equation

$$f_1(\mathbf{Z}_1, \mathbf{Z}_2) + f_2(\mathbf{Z}_2, \mathbf{Z}_3) + f_3(\mathbf{Z}_3, \mathbf{Z}_4) + f_4(\mathbf{Z}_4, \mathbf{Z}_1) = \mathbf{O},$$

which is a particular case for $n = 4$ and $p = 2$ of equation (1), is given by

$$\begin{aligned}
f_1(\mathbf{Z}_1, \mathbf{Z}_2) &= F_{12}(\mathbf{Z}_2) - A_{13} + F_{14}(\mathbf{Z}_1),\\
f_2(\mathbf{Z}_2, \mathbf{Z}_3) &= -F_{12}(\mathbf{Z}_2) - F_{23}(\mathbf{Z}_3) + A_{24},\\
f_3(\mathbf{Z}_3, \mathbf{Z}_4) &= A_{13} + F_{23}(\mathbf{Z}_3) + F_{34}(\mathbf{Z}_4),\\
f_4(\mathbf{Z}_4, \mathbf{Z}_1) &= -F_{14}(\mathbf{Z}_1) - A_{24} - F_{34}(\mathbf{Z}_4),
\end{aligned}$$

where F_{ij} $(1 \le i \le 3;\ 2 \le j \le 4)$ are arbitrary complex vector functions from $\mathcal{V}$ and A_{ij} $(i = 1, 2;\ j = 3, 4)$ are arbitrary constant complex vectors also from $\mathcal{V}$.

Example 2. Consider the functional equation

$$f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f_3(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) + f_4(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}, \qquad (10)$$

where $f_i \colon \mathcal{V}^3 \to \mathcal{V}$ $(1 \le i \le 4)$. This equation is a particular case for $p = 3$ and $n = 4$ of functional equation (1).

According to Theorem 1, the general solution of functional equation (10) is

$$f_1(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = F_{12}(\mathbf{Z}_2,\mathbf{Z}_3) - F_{13}(\mathbf{Z}_1,\mathbf{Z}_3) + F_{14}(\mathbf{Z}_1,\mathbf{Z}_2), \tag{11}$$
$$f_2(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = -F_{12}(\mathbf{Z}_2,\mathbf{Z}_3) - F_{23}(\mathbf{Z}_3,\mathbf{Z}_4) + F_{24}(\mathbf{Z}_2,\mathbf{Z}_4), \tag{12}$$
$$f_3(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) = F_{13}(\mathbf{Z}_1,\mathbf{Z}_3) + F_{23}(\mathbf{Z}_3,\mathbf{Z}_4) + F_{34}(\mathbf{Z}_1,\mathbf{Z}_4), \tag{13}$$
$$f_4(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) = -F_{14}(\mathbf{Z}_1,\mathbf{Z}_2) - F_{24}(\mathbf{Z}_2,\mathbf{Z}_4) - F_{34}(\mathbf{Z}_1,\mathbf{Z}_4), \tag{14}$$

where F_{12}, F_{13}, F_{14}, F_{23} and F_{34} are arbitrary complex vector functions from $\mathcal{V}$.

Theorem 2. The general solution of complex vector functional equation (2) is given by

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_p) \tag{15}$$

$$= \sum_{i=1}^{n} \alpha_i \Bigg[\sum_{j=1}^{i-1} (-1)^{i+1}\mathcal{C}^{n+1-i} G_{ji}(\{\mathbf{Z}_i,\mathbf{Z}_{i+1},\ldots,\mathbf{Z}_{i+p-1}\}\cap\{\mathbf{Z}_j,\mathbf{Z}_{j+1},\ldots,\mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{j=i+1}^{n} (-1)^{j}\mathcal{C}^{n+1-i} G_{ij}(\{\mathbf{Z}_i,\mathbf{Z}_{i+1},\ldots,\mathbf{Z}_{i+p-1}\}\cap\{\mathbf{Z}_j,\mathbf{Z}_{j+1},\ldots,\mathbf{Z}_{j+p-1}\})\Bigg],$$

where $\mathcal{C}$ is a cyclic operator such that $\mathcal{C}G(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) = G(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_n,\mathbf{Z}_1)$; G_{ij} $(1 \le i \le n-1;\ i+1 \le j \le n)$ are arbitrary complex vector functions from $\mathcal{V}$ and $\sum_a^s = \mathbf{O}$ $(a > s)$.

Proof. If we apply operator $\mathcal{C}^{n+1-i}$ to both sides of (3) (with f_i replaced by $a_i f$) and then multiply by indefinite complex constants α_i $(1 \le i \le n)$, we have

$$a_i\alpha_i\mathcal{C}^{n+1-i} f(\mathbf{Z}_i,\mathbf{Z}_{i+1},\ldots,\mathbf{Z}_{i+p-1}) \tag{16}$$

$$= \alpha_i \Bigg[\sum_{j=1}^{i-1} (-1)^{i+1}\mathcal{C}^{n+1-i} F_{ji}(\{\mathbf{Z}_i,\mathbf{Z}_{i+1},\ldots,\mathbf{Z}_{i+p-1}\}\cap\{\mathbf{Z}_j,\mathbf{Z}_{j+1},\ldots,\mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{j=i+1}^{n} (-1)^{j}\mathcal{C}^{n+1-i} F_{ij}(\mathbf{Z}_i,\mathbf{Z}_{i+1},\ldots,\mathbf{Z}_{i+p-1}\}\cap\{\mathbf{Z}_j,\mathbf{Z}_{j+1},\ldots,\mathbf{Z}_{j+p-1}\})\Bigg]$$

$$(1 \le i \le n).$$

By summing up the above functions (16), we obtain

$$\left(\sum_{i=1}^{n} a_i\alpha_i\right) f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) \tag{17}$$

$$= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{i-1} (-1)^{i+1} \mathcal{C}^{n+1-i} F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \right.$$

$$\left. + \sum_{j=i+1}^{n} (-1)^j \mathcal{C}^{n+1-i} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \right].$$

From equation (2) and equality (17) we find

$$\mathbf{O} = \sum_{r=1}^{n} a_r \left(\sum_{i=1}^{n} a_i\alpha_i\right) \mathcal{C}^{r-1} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p)$$

$$= \sum_{r=1}^{n} a_r \left\{ \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{i-1} (-1)^{i+1} \mathcal{C}^{n+r-i} \right.\right.$$

$$F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1} \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{j=i+1}^{n-1} (-1)^j \mathcal{C}^{n+r-i}$$

$$\left.\left. F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \dots, \mathbf{Z}_{j+p-1}\}) \right] \right\}$$

$$= \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+1} \alpha_i \sum_{r=1}^{n} a_r \mathcal{C}^{n+r-i}$$

$$F_{ji}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (-1)^{j} \alpha_i \sum_{r=1}^{n} a_r \mathcal{C}^{n+r-i}$$

$$F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (-1)^{j} \bigg\{ \sum_{r=1}^{n} \bigg[\alpha_i \mathcal{C}^{n+r-i}$$

$$F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$- \alpha_j \mathcal{C}^{n+r-j} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\}) \bigg] \bigg\}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (-1)^{j} \mathcal{C}^{n+1-i} \bigg\{ \sum_{r=1}^{n} a_r \bigg[\alpha_i \mathcal{C}^{r-1}$$

$$F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\})$$

$$- \alpha_j \mathcal{C}^{r-1} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_{2i-j}, \mathbf{Z}_{2i-j+1}, \ldots, \mathbf{Z}_{2i-j+p-1}\}) \bigg] \bigg\}.$$

Now, we may determine the constants α_i $(1 \leq i \leq n)$ from the identities

$$\sum_{r=1}^{n} a_r \bigg[\alpha_i \mathcal{C}^{r-1} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\}) \tag{18}$$

$$- \alpha_j \mathcal{C}^{r-1} F_{ij}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}\} \cap \{\mathbf{Z}_{2i-j}, \mathbf{Z}_{2i-j+1}, \ldots, \mathbf{Z}_{2i-j+p-1}\}) \bigg] = \mathbf{O}$$

$$(1 \leq i \leq n-1; \quad i+1 \leq j \leq n).$$

Equation (17) may be rewritten in the following form

$$\left(\sum_{i=1}^{n} a_i\alpha_i\right)T = S, \tag{19}$$

and because it has a unique solution T, by introducing the new functions

$$\frac{F_{ij}}{\sum_{i=1}^{n} a_i\alpha_i} = G_{ij} \qquad (1 \leq i \leq n-1; \quad i+1 \leq j \leq n) \tag{20}$$

(15) follows.

Example 3. We consider the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) - f(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}. \tag{21}$$

According to (11), (12), (13) and (14) we obtain

$$\begin{aligned}
&f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = F_{12}(\mathbf{Z}_2, \mathbf{Z}_3) - F_{13}(\mathbf{Z}_1, \mathbf{Z}_3) + F_{14}(\mathbf{Z}_1, \mathbf{Z}_2),\\
&-f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = -F_{12}(\mathbf{Z}_1, \mathbf{Z}_2) - F_{23}(\mathbf{Z}_2, \mathbf{Z}_3) + F_{24}(\mathbf{Z}_1, \mathbf{Z}_3),\\
&f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = F_{13}(\mathbf{Z}_3, \mathbf{Z}_1) + F_{23}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{34}(\mathbf{Z}_3, \mathbf{Z}_2),\\
&-f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = -F_{14}(\mathbf{Z}_2, \mathbf{Z}_3) - F_{24}(\mathbf{Z}_3, \mathbf{Z}_1) - F_{34}(\mathbf{Z}_2, \mathbf{Z}_1).
\end{aligned}$$

On the basis of the identities in (18), we determine the constants α_i ($i = 1, 2, 3, 4$) as follows

$$\begin{aligned}
&\alpha_1 F_{12}(\mathbf{Z}_2, \mathbf{Z}_3) - \alpha_2 F_{12}(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_1 F_{12}(\mathbf{Z}_3, \mathbf{Z}_4) + \alpha_2 F_{12}(\mathbf{Z}_2, \mathbf{Z}_3)\\
&+ \alpha_1 F_{12}(\mathbf{Z}_4, \mathbf{Z}_1) - \alpha_2 F_{12}(\mathbf{Z}_3, \mathbf{Z}_4) - \alpha_1 F_{12}(\mathbf{Z}_1, \mathbf{Z}_2) + \alpha_2 F_{12}(\mathbf{Z}_3, \mathbf{Z}_1) = \mathbf{O},
\end{aligned}$$

$$\begin{aligned}
&-\alpha_1 F_{13}(\mathbf{Z}_1, \mathbf{Z}_3) + \alpha_3 F_{13}(\mathbf{Z}_3, \mathbf{Z}_1) + \alpha_1 F_{13}(\mathbf{Z}_2, \mathbf{Z}_4) - \alpha_3 F_{13}(\mathbf{Z}_4, \mathbf{Z}_2)\\
&-\alpha_1 F_{13}(\mathbf{Z}_3, \mathbf{Z}_1) + \alpha_3 F_{13}(\mathbf{Z}_1, \mathbf{Z}_3) + \alpha_1 F_{13}(\mathbf{Z}_4, \mathbf{Z}_2) - \alpha_3 F_{13}(\mathbf{Z}_2, \mathbf{Z}_4) = \mathbf{O},
\end{aligned}$$

$$\begin{aligned}
&\alpha_1 F_{14}(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_4 F_{14}(\mathbf{Z}_2, \mathbf{Z}_3) - \alpha_1 F_{14}(\mathbf{Z}_2, \mathbf{Z}_3) + \alpha_4 F_{14}(\mathbf{Z}_3, \mathbf{Z}_4)\\
&+ \alpha_1 F_{14}(\mathbf{Z}_3, \mathbf{Z}_4) - \alpha_4 F_{14}(\mathbf{Z}_4, \mathbf{Z}_1) - \alpha_1 F_{14}(\mathbf{Z}_4, \mathbf{Z}_1) + \alpha_4 F_{14}(\mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O},
\end{aligned}$$

$$
\begin{aligned}
&- \alpha_2 F_{23}(\mathbf{Z}_2, \mathbf{Z}_3) + \alpha_3 F_{23}(\mathbf{Z}_1, \mathbf{Z}_2) + \alpha_2 F_{23}(\mathbf{Z}_3, \mathbf{Z}_4) - \alpha_3 F_{23}(\mathbf{Z}_2, \mathbf{Z}_3)\\
&- \alpha_2 F_{23}(\mathbf{Z}_4, \mathbf{Z}_1) + \alpha_3 F_{23}(\mathbf{Z}_3, \mathbf{Z}_4) + \alpha_2 F_{23}(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_3 F_{23}(\mathbf{Z}_4, \mathbf{Z}_1) = \mathbf{O},
\end{aligned}
$$

$$
\begin{aligned}
&\alpha_2 F_{24}(\mathbf{Z}_1, \mathbf{Z}_3) - \alpha_4 F_{24}(\mathbf{Z}_3, \mathbf{Z}_1) - \alpha_2 F_{24}(\mathbf{Z}_2, \mathbf{Z}_4) + \alpha_4 F_{24}(\mathbf{Z}_4, \mathbf{Z}_2)\\
&+ \alpha_2 F_{24}(\mathbf{Z}_3, \mathbf{Z}_1) - \alpha_4 F_{24}(\mathbf{Z}_1, \mathbf{Z}_3) - \alpha_2 F_{24}(\mathbf{Z}_4, \mathbf{Z}_2) + \alpha_4 F_{24}(\mathbf{Z}_1, \mathbf{Z}_4) = \mathbf{O},
\end{aligned}
$$

$$
\begin{aligned}
&\alpha_3 F_{34}(\mathbf{Z}_3, \mathbf{Z}_2) - \alpha_4 F_{34}(\mathbf{Z}_2, \mathbf{Z}_1) - \alpha_3 F_{34}(\mathbf{Z}_4, \mathbf{Z}_3) + \alpha_4 F_{34}(\mathbf{Z}_3, \mathbf{Z}_2)\\
&+ \alpha_3 F_{34}(\mathbf{Z}_1, \mathbf{Z}_4) - \alpha_4 F_{34}(\mathbf{Z}_4, \mathbf{Z}_3) - \alpha_3 F_{34}(\mathbf{Z}_2, \mathbf{Z}_1) + \alpha_4 F_{34}(\mathbf{Z}_1, \mathbf{Z}_4) = \mathbf{O}.
\end{aligned}
$$

Thus, we obtain

$$\alpha_1 = -\alpha_2, \quad \alpha_1 = \alpha_3, \quad \alpha_1 = -\alpha_4, \quad \alpha_2 = -\alpha_3, \quad \alpha_2 = \alpha_4, \quad \alpha_3 = -\alpha_4,$$

which means that $\alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4 = 1$.

The general solution of functional equation (21) is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = F(\mathbf{Z}_1, \mathbf{Z}_2) + F(\mathbf{Z}_2, \mathbf{Z}_3) + G(\mathbf{Z}_1, \mathbf{Z}_3) - G(\mathbf{Z}_3, \mathbf{Z}_1),$$

where

$$
\begin{aligned}
&F(\mathbf{Z}_1, \mathbf{Z}_2) = F_{14}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{12}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{23}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{34}(\mathbf{Z}_2, \mathbf{Z}_1),\\
&G(\mathbf{Z}_1, \mathbf{Z}_3) = -F_{13}(\mathbf{Z}_1, \mathbf{Z}_3) - F_{24}(\mathbf{Z}_1, \mathbf{Z}_3).
\end{aligned}
$$

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