INDEX OF A SUBGROUP OF AN ABELIAN GROUP

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Introduction. Lemma 1 of [2] states that if an abelian group G is a union of cosets $H_1 + a_1$, $H_2 + a_2$, $\ldots$, $H_n + a_n$, where the H_i's are subgroups of G and the a_i's are elements of G, then the index of one of these subgroups is finite. In that paper nothing was said about an upper bound for the index of that subgroup. In this paper, under the hypothesis of Lemma 1 of [2], we show that, for $n \geq 3$, the index of one of those subgroups is at most $2^{2(n-2)} - 2^{n-2} + 1$. The above conclusion is not true for nonabelian groups. It is easy to see that if

$$G = \bigcup_{i=1}^{2} (H_i + a_i)$$

and the H_i's are proper subgroups of G, then $H_1 = H_2$. Further, we show that if

$$G = \bigcup_{i=1}^{3} (H_i + a_i)$$

and the H_i's are proper subgroups of G, then either the index of one of these subgroups is two or all these subgroups are identical, and hence the index of one of these subgroups is at most three.

Notation. If A and B are subsets of an abelian group $(G, +)$, then $A + B$ and $A - B$ stand for $\{x + y : x \in A \text{ and } y \in B\}$ and $\{x - y : x \in A \text{ and } y \in B\}$, respectively. For $g \in G$, $A + \{g\}$ is denoted by $A + g$. The set $A \setminus B$ denotes $\{x \in A : x \notin B\}$.

Lemma 1. Let a and b be elements of an abelian group $(G, +)$. Suppose that H and K are proper subgroups of G such that H is not contained in K, and K is not contained in H. Then there exists a subset M of G of cardinality four such that, for every $g \in G$, the intersection of the set $M + g$ and the set $G \setminus ((H + a) \cup (K + b))$ is nonempty.

Proof. Let $h \in H \setminus K$, $k \in K \setminus H$, and $M = \{0,\ h,\ k,\ h + k\}$. Suppose, to the contrary, that $M + g \subseteq (H + a) \cup (K + b)$ for some $g \in G$. Without loss of generality, we may assume that $g \in H + a$. If $k + g \in H + a$, then $k = (k + g) - g \in$

$(H+a)-(H+a)=H$, which contradicts that $k \in K \setminus H$. Hence, $k+g \in K+b$. If $h+k+g \in K+b$, then $h=(h+k+g)-(k+g) \in (K+b)-(K+b)=K$, which contradicts that $h \in H \setminus K$. Hence, $h+k+g \in H+a$. Now $k=(h+k+g)-(h+g) \in (H+a)-(H+a)=H$, which is a contradiction. Thus, for every $g \in G$, $(M+g) \cap (G \setminus ((H+a) \cup (K+b))$ is nonempty.

Lemma 2. If H and K are proper subgroups of an abelian group $(G,+)$ and $G=(H+a) \cup (K+b)$ for some elements a and b of G, then $H=K$.

Proof. By Lemma 1, $H \subseteq K$ or $K \subseteq H$. Without loss of generality, we may assume that $H \subseteq K$. Suppose that H is properly contained in K. Let $k \in K \setminus H$. Then $k+a \notin H+a$. Hence, $k+a \in K+b$. Since the cosets of a subgroup are either identical or disjoint, $K+a=K+b$. Consequently, $G=(H+a) \cup (K+a)$ and $G=G-a=H \cup K$, which contradicts the fact that no group is the union of two of its proper subgroups.

Lemma 3. If H, K, and S are proper subgroups of an abelian group $(G,+)$ and $G=H \cup K \cup S$, then

(i) the intersection of any two of the subgroups H, K, S is the same as the intersection of all the subgroups H, K, S;

(ii) the index of each of the subgroups H, K, S is two.

Proof of (i). If S is contained in $H \cup K$, then $G=H \cup K$, which contradicts the fact that no group is the union of two of its proper subgroups. Let $s \in S \setminus (H \cup K)$ and $x \in H \cap K$. Then $x+s \notin H \cup K$. Hence, $x+s \in S$ and $x \in S-s=S$. This shows that $H \cap K \subseteq S$ and hence, $H \cap K = H \cap K \cap S$. Similarly, $H \cap S = K \cap S = H \cap K \cap S$.

Proof of (ii). Since no group is the union of two of its proper subgroups, $H \setminus S$ and $K \setminus S$ are nonempty. Let $h \in H \setminus S$ and $k \in K \setminus S$. Then, since $H \cap K = H \cap K \cap S$, we have $h \notin K$ and $k \notin H$. It is easy to see that $h+k \notin H \cup K$.

Hence,

$$h+k \in S \text{ for all } h \in H \setminus S \text{ and } k \in K \setminus S. \tag{$*$}$$

This implies that $\emptyset \neq H \setminus S \subseteq S-k$ for all $k \in K \setminus S$ and $\emptyset \neq K \setminus S \subseteq S-h$ for all $h \in H \setminus S$. Let k be a fixed element of $K \setminus S$. Note that $k \in K \setminus S$ if and only if $-k \in K \setminus S$. Now, it follows from $(*)$ that $h-k \in S$ for all $h \in H \setminus S$, and hence, $S-h=S-k$ for all $h \in H \setminus S$. Consequently, $H \setminus S \subseteq S-k$ and $K \setminus S \subseteq S-k$. Now since $G=H \cup K \cup S=(H \setminus S) \cup (K \setminus S) \cup S=(S-k) \cup S$, the index of S is two. Similarly, the index of each of the subgroups H, K is two.

Remark 1. There exists an abelian group G, and three distinct proper subgroups H, K, S of G such that no two elements of $\{H,K,S\}$ are linearly ordered under set inclusion (i.e., $X \not\subset Y$ and $Y \not\subset X$ for $X,Y \in \{H,K,S\}$ and $X \neq Y$) and $G=H \cup K \cup S$.

For example, let $G = U(8) = \{1, 3, 5, 7\}$, $H = \{1, 3\}$, $K = \{1, 5\}$, and $S = \{1, 7\}$.

However, it follows from the proof of the following theorem that in the above statement if $G = H \cup K \cup S$ is replaced by $G = H \cup K \cup (S + c)$ for some $c \in G \setminus S$, then we have $S \subseteq H$ or $S \subseteq K$.

Theorem 1. Let H, K, and S be proper subgroups of an abelian group $(G, +)$ and $G = H \cup K \cup (S + c)$ for some $c \in G$. Then

(i) the index of two of the subgroups H, K, S is two if $c \notin S$;
(ii) the index of each of the subgroups H, K, S is two if $c \in S$.

Proof of (i). Since the set $(S+c) \cap S$ is empty, S is contained in the set $H \cup K$. Hence, $S = (S \cap H) \cup (S \cap K)$. Since no group is the union of two of its proper subgroups, $S \cap H = S$ or $S \cap K = S$. Without loss of generality we may assume that $S \cap H = S$. This is equivalent to $S \subseteq H$.

Case 1. $H \subseteq K$.

In this case $G = K \cup (S+c)$. Now, by Lemma 2, $K = S$. Since $S \subseteq H \subseteq K = S$, $H = K = S$ and the index of each of these subgroups is two.

Case 2. $K \subseteq H$.

In this case $G = H \cup (S + c)$. Now, by Lemma 2, $H = S$ and the index of H is two.

Case 3. H is not contained in K, and K is not contained in H.

Let $h \in H \setminus K$ and $k \in K \setminus H$. Then $h+k \notin H \cup K$ and hence, $h+k \in S+c$. Let h be a fixed element of $H \setminus K$. Then $K \setminus H \subseteq S+c-h$ and $G = H \cup (K \setminus H) \cup (S+c) = H \cup (S+c-h) \cup (S+c)$. Since $S \subseteq H$ and $H+c-h = H+c$, we get $G = H \cup (H+c)$ and the index of H is two. Furthermore, in this case, we shall show that the index of one of the subgroups K, S is two. If S is contained in K, then, as done above for $S \subseteq H$, we have $G = K \cup (K + c)$, and hence, the index of K is two.

If S is not contained in K, then we shall show that $S = H$ and the index of S is two. To verify this, let $s \in S \setminus K$ and $x \in K$. Then $x + s \notin K$ and hence, $x + s \in H \cup (S + c)$. Since $s \in S \subseteq H$, we get $x = (x + s) - s \in (H \cup (S + c)) - s = (H - s) \cup (S + c - s) = H \cup (S + c)$ and hence, $K \subseteq H \cup (S + c)$. Now, since $G = H \cup (S + c)$, by Lemma 2, $H = S$ and the index of S is two. This completes the proof of Case 3.

Thus, in all three cases, the index of two of the subgroups H, K, S is two.

Part (ii) of Theorem 1 is a restatement of Lemma 3(ii).

Corollary 1. Suppose H, K, and S are proper subgroups of an abelian group $(G, +)$ such that H is not contained in K, K is not contained in H, and $G = H \cup K \cup (S + c)$ for some $c \in G \setminus S$. Then

(i) if S is not contained in $H \cap K$, $S = H$ or $S = K$, and the index of S is two.

(ii) if S is contained in $H \cap K$, H and K have the same index two and the index of S is at most four.

Proof of (i). We proved in Theorem 1 that $S \subseteq H$ or $S \subseteq K$, and if S is not contained in K, then $S = H$ and the index of S is two; similarly, if S is not contained in H, then $S = K$ and the index of S is two.

Proof of (ii). If $H \subseteq K$ or $K \subseteq H$, then it follows from the proofs of Case 1 and Case 2 that $H = K = S$ and the index of each of the subgroups is two. Suppose H is not contained in K, and K is not contained in H. By Case 3 of the previous theorem, if $S \subseteq H$, then the index of H is two. Similarly, if $S \subseteq K$, then the index of K is two. By Lemma 1, there exists a set M of cardinality four such that $(M + g) \cap (G \setminus (H \cup K)$ is nonempty for every g in G. Hence, since $G \setminus (H \cup K) \subseteq S + c$, we have $(M + g) \cap (S + c) \neq \emptyset$ for all $g \in G$. This shows that $G = S + c - M$ and the index of S is at most four.

Remark 2. In Corollary 1, if $c \in S$ instead of $c \in G \setminus S$, then S is not contained in $H \cap K$; otherwise $G = H \cup K$, which is impossible. Further, it is not true that $S = H$ or $S = K$. For example, let $G = U(8) = \{1, 3, 5, 7\}$, $H = \{1, 3\}$, $K = \{1, 5\}$, and $S = \{1, 7\}$.

From Corollary 1 and the proof of Theorem 1, we obtain the following corollary.

Corollary 2. In Theorem 1(i), if G is not equal to the proper subunion (i.e., $H \cup (S + c) \neq G \neq K \cup (S + c)$), then $S \subseteq H \cap K$, H and K have the same index two, and the index of S is at most four.

The following example shows that in Corollary 2, the upper bound four for the index of S cannot be improved.

Example 1. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H = \mathbb{Z}_4 \oplus \{0\} \oplus \mathbb{Z}_2$, $K = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \{0\}$, $S = \mathbb{Z}_4 \oplus \{0\} \oplus \{0\}$ and $c = (0, 1, 1)$. Then $G = H \cup K \cup (S + c)$, H and K have the same index two, and the index of S is four.

Theorem 2. If H, K, and S are proper subgroups of an abelian group $(G, +)$ and $G = H \cup (K + b) \cup (S + c)$ for some elements b and c of G, then either the index of one of the subgroups H, K, S is two or $H = K = S$.

Proof. It is sufficient to show that if the index of each of the subgroups H, K, S is not equal to two, then $H = K = S$. Suppose that the index of each of the subgroups H, K, S is not equal to two.

Claim 1. $b \notin H + K$ and $c \notin H + S$.

Proof. If $b \in H + K$, then $b = h + k$ for some $h \in H$ and $k \in K$. Since $H - h = H$, $K - k = K$, and $G = H \cup (K + b) \cup (S + c)$, we have $G = G - h = (H - h) \cup (K + b - h) \cup (S + c - h) = H \cup K \cup (S + c - h)$. Theorem 1 yields that the index of two of the subgroups H, K, S is two, which contradicts our assumption. Hence, $b \notin H + K$. Similarly, $c \notin H + S$.

Claim 2. $K = S$.

Proof. If $(K + b) \setminus (S + c)$ is empty, then, since $G = H \cup (K + b) \cup (S + c)$, we have $G = H \cup (S + c)$. Consequently, by Lemma 2, the index of two of the subgroups H, K, S is two. Suppose that $(K + b) \setminus (S + c)$ is nonempty. Let $x \in (K + b) \setminus (S + c)$. We shall prove that S is contained in K. The proof of K is contained in S is similar. Suppose $S \setminus K \neq \emptyset$. Let $s \in S \setminus K$. Then it is easy to see that $x+s \notin S+c$. $x+s \notin K+b$; otherwise $s = (x+s)-x \in (K+b)-(K+b) = K$, which contradicts that $s \in S \setminus K$. Consequently, $x + s \in H$. This shows that $x \in H - s$ and $(K + b) \setminus (S + c) \subseteq H - s$, and hence,

$$G = H \cup (H - s) \cup (S + c). \tag{$\bullet$}$$

Let $h \in H$. By Claim 1, $c \notin H + S$. Hence, $h + c \notin H$ and $h + c \notin H - s$. By ($\bullet$), $h + c \in S + c$ and hence, $h \in S$. This implies that $H \subseteq S$. Consequently, by ($\bullet$) and the fact that $S - s = S$, we have $G = S \cup (S - s) \cup (S + c) = S \cup (S + c)$, which contradicts our assumption that the index of S is not equal to two. Thus $S \subseteq K$. Similarly, $K \subseteq S$.

Claim 3. $H = K$.

Proof. By Claim 1, we have $b \notin H + K$, $c \notin H + S$, and, by Claim 2, $K = S$. Since the cosets of a subgroup are either identical or disjoint, we have $K \cap (K + b) = \emptyset = K \cap (K + c)$. Note that $G = H \cup (K + b) \cup (K + c)$. Hence, $K \subseteq H$.

To show that $H = K$, suppose that K is properly contained in H. Let $h \in H \setminus K$. Then $h + b \notin K + b$. Since $b \notin H$, $h + b \notin H$. Consequently, $h + b \in K + c$. This implies that $b - c \in K - h \subseteq H$. Hence, $H + b = H + c$. Now $G = H \cup (K + b) \cup (K + c) = H \cup (H + b) \cup (H + c) = H \cup (H + b)$ and hence, the index of H is two, which contradicts our assumption. This completes the proof of Claim 3. Thus, $H = K = S$.

Corollary 3. In Theorem 2, if the index of H is two and $K \cup S$ is not contained in H, then the index of K or S is two.

Proof. If $K \cup S$ is not contained in H, then K is not contained in H or S is not contained in H. Without loss of generality, we may assume that K is not contained in H. If $H \subseteq K$, then $|G : K| < |G : H| = 2$, which contradicts that K is a proper subgroup of G. So, H is not contained in K. Since the index of H is two and H is properly contained in the subgroup $H + K$, we get $H + K = G$. Since $b \in H + K$, the proof of Claim 1 of Theorem 2 shows that $G = H \cup K \cup (S + c - h)$. Now, the desired result follows from Theorem 1.

Corollary 4. If H, K, and S are proper subgroups of an abelian group $(G,+)$ and $G=(H+a)\cup(K+b)\cup(S+c)$ for some elements a, b, c of G, then either one of the subgroups H, K, S has index two or $H=K=S$.

Proof. Since $0\in G$, we have $H+a=H$, $K+b=K$, or $S+c=S$. Now, the corollary follows from Theorem 2.

Remark 3. It is interesting to note that Theorem 2 is not true for nonabelian groups. Let G be the symmetric group S_3, $H=\{(1),(1,2)\}$, $K=\{(1),(1,3)\}$, $S=\{(1),(2,3)\}$, $b=(1,3,2)$ and $c=(1,3)$. Then H, K, and S are distinct subgroups of G, and $G=H\cup(K+b)\cup(S+c)$ for some $b,c\in G$, but the index of each of the subgroups H, K, S is three.

The following theorem is a generalization of a result in [1,2]. Lemma 1 of [2] states that if an abelian group $(G,+)$ is the set theoretic union of finitely many cosets,

$$G=\bigcup_{i=1}^{n}(H_i+a_i),$$

where $a_i\in G$, and the H_i's are subgroups of G, then the index of H_i is finite for some i.

Theorem 3. If the H_i's are proper subgroups of an abelian $(G,+)$ and G is the set-theoretic union of finitely many cosets,

$$G=\bigcup_{i=1}^{n}(H_i+a_i),$$

where $a_i\in G$ and $n\geq 4$, then the index of H_i is at most $2^{2(n-2)}-2^{(n-2)}+1$ for at least two values of i.

We prove first the following two lemmas.

Lemma 4. Let M be a subset of an abelian group $(G,+)$ of cardinality 2^{n-1}, where $n\geq 4$, and let $a\in G$. Suppose that the index of a subgroup H of G is greater than $2^{2(n-2)}-2^{(n-2)}+1$. Then there exists a $g\in G$ such that $M+g\subseteq G\setminus(H+a)$.

Proof. If, for every $g\in G$, $M+g$ is not contained in $G\setminus(H+a)$, then $(M+g)\cap(H+a)$ is nonempty for every $g\in G$, and hence, $G=H+a-M$, which implies that the index of H is at most $|a-M|=2^{n-1}$. But this together with the fact that $2^{n-1}<2^{2(n-2)}-2^{n-2}+1$ for all $n\geq 4$ contradicts the hypothesis that the index of H is greater than $2^{2(n-2)}-2^{(n-2)}+1$.

Lemma 5. Suppose that the index of a subgroup H of an abelian group $(G,+)$ is greater than $2^{2(n-2)} - 2^{(n-2)} + 1$, where $n \geq 4$. Let S be a subset of G that has the following property: for every subset X of G of cardinality 2^m, where m is a positive integer and $m \leq n-1$, there exists a $g \in G$ such that $X + g \subseteq S$. Let $a \in G$. Then $S \setminus (H+a)$ has the following property: for every subset Y of G of cardinality 2^{m-1}, there exists $g \in G$ such that $Y + g \subseteq S \setminus (H+a)$.

Proof. Let Y be a subset of G of cardinality 2^{m-1}, $1 \leq m \leq n-1$, and $g \in G$. Then, since the cardinality of the set $Y \cup (Y+g)$ is at most 2^m, $(Y \cup (Y+g)) + r \subseteq S$ for some $r \in G$. Suppose that the conclusion of the lemma is false. Then $(Y+r) \cap (H+a) \neq \emptyset$ and $(Y+(g+r)) \cap (H+a) \neq \emptyset$. Consequently, $g = (g+r) - r \in (H+a-Y) - (H+a-Y) = H - H + Y - Y = H + Y - Y$. Since $g \in H + (Y-Y)$ for all $g \in G$, $|Y-Y| \leq |Y|^2 - |Y| + 1 = 2^{2(m-1)} - 2^{m-1} + 1$, and $m \leq n-1$, the index of H is at most $2^{2(m-1)} - 2^{m-1} + 1 \leq 2^{2(n-2)} - 2^{n-2} + 1$, which contradicts the hypothesis that the index of H is greater than $2^{2(n-2)} - 2^{(n-2)} + 1$.

Proof of Theorem 3. Assume that the conclusion of the theorem is false. Then, for at least $n-1$ values of i, the index of H_i is greater than $2^{2(n-2)} - 2^{(n-2)} + 1$. Without loss of generality, we may assume that the index of H_i is greater than $2^{2(n-2)} - 2^{(n-2)} + 1$ for each i, $1 \leq i \leq n-1$. Then, by Lemma 4, $G \setminus (H_1 + a_1)$ contains a translated copy of every set of cardinality 2^{n-1}. Applying Lemma 5 to the set $G \setminus (H_1 + a_1)$, we get

$$(G \setminus (H_1 + a_1)) \setminus (H_2 + a_2) = G \setminus \bigcup_{i=1}^{2} (H_i + a_i)$$

contains a translated copy of every set of cardinality 2^{n-2}. Continuing in this fashion, we obtain that

$$G \setminus \bigcup_{i=1}^{n-1} (H_i + a_i)$$

contains a translated copy of every set of cardinality $2^{n-(n-1)} = 2$. Since

$$G \setminus \bigcup_{i=1}^{n-1} (H_i + a_i) \subseteq (H_n + a_n)$$

and

$$G \setminus \bigcup_{i=1}^{n-1} (H_i + a_i)$$

contains a translated copy of every set of cardinality 2, so does the set $H_n + a_n$. Let $g \in G \setminus H_n$. Then $\{0, g\} + x \subseteq H_n + a_n$ for some $x \in G$, which implies that $g = (g + x) - x \in H_n + a_n - (H_n + a_n) = H_n$. This contradicts that H_n is a proper subgroup of G. Thus the proof of the theorem is complete.

Remark 4. Clearly, Theorem 3 is not true for $n = 2$. To see that Theorem 3 is not true for $n = 3$, let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $H_1 = \mathbb{Z}_4 \oplus \{0\}$, $H_2 = H_1$, $H_3 = \mathbb{Z}_4 \oplus \{0, 2\}$, $a_1 = (0, 1)$, $a_2 = (0, 3)$ and $a_3 = (0, 0)$.

References

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