A CHARACTERIZATION OF METACOMPACTNESS IN TERMS OF FILTERS

James E. Joseph, Myung H. Kwack, and Bhamini M. P. Nayar

Elementary general topology courses customarily include characterizations of compact topological spaces, Lindelöf spaces and countably compact spaces in terms of filters and filterbases. The purpose of this note is to add to the literature a characterization of metacompactness in terms of filters. The reader is referred to Bourbaki [2] for definitions and results used but not given here. All spaces in this note are topological spaces. However, no separation axioms are assumed unless explicitly stated. A collection of subsets of a set is *point–finite* if each point in the set is a member of at most finitely many members of the collection. In this article we will say that a family of sets Ω *refines a family of sets* Γ if each $A \in \Omega$ satisfies $A \subset B$ for some $B \in \Gamma$ and $\cup_{\Omega} A = \cup_{\Gamma} A$. A topological space is defined to be *metacompact* if every covering by open sets has a point–finite open refinement (see [1]). It is proved in [1] that a T_1 countably compact metacompact space is compact. Here we 1) characterize metacompactness by filters, 2) use the characterization to give two proofs that a countably compact metacompact space is compact, 3) give a relationship between continuity of functions and filters characterizating metacompactness and some new proofs of known results to illustrate how this relationship may be used to shorten proofs of results involving continuity and metacompactness.

Definition 1. A collection of subsets Ω of a set X is *point dominating (p.d.)* if each $x \in X$ is a member of all but finitely many elements of Ω.

Definition 2. A filter on a space is of type $\mathcal{M}$ if every p.d. subcollection of the filter has nonempty adherence.

We give the following lemma without proof.

Lemma 1. A filter on a space is of type $\mathcal{M}$ if and only if every closed p.d. subcollection of the filter has nonempty adherence.

The following theorem gives the promised characterization of metacompactness by filters.

Theorem 1. A topological space is metacompact if and only if every filter of type $\mathcal{M}$ on the space has nonempty adherence.

Proof. For the necessity proof, let the space X be metacompact and let Ω be a filter on X such that $adh\Omega$ is empty. Then $\{X - \overline{F} : F \in \Omega\}$ is an open cover of X and thus, has an open point–finite refinement κ. Let $\kappa^* = \{X - G : G \in \kappa\}$. Then $\kappa^* \subset \Omega$ since if G is an element of κ, some $F \in \Omega$ satisfies $G \subset X - \overline{F} \subset X - F$ so $F \subset X - G$. Also, it is clear that κ^* is p.d. since κ is point–finite and that adh $\kappa^* = \cap_{G \in \kappa}(X - G) = \emptyset$. Thus, Ω is not of type $\mathcal{M}$. For the sufficiency proof, suppose every filter of type $\mathcal{M}$ on the space has nonempty adherence, and let Ω be an open cover of X with no finite subcover. Then $\{X - \cup_\Gamma A : \Gamma \subset \Omega, \Gamma \text{ finite}\}$ is a base for a filter on X with empty adherence which, from Lemma 1, has a p.d. subcollection Λ consisting of closed subsets with empty adherence; then $\{X - F : F \in \Lambda\}$ is a point–finite collection of open sets. For each $F \in \Lambda$ choose a finite $\Omega(F) \subset \Omega$ such that $X - \cup_{\Omega(F)} A \subset F$, so $X - F \subset \cup_{\Omega(F)} A$. For $F \in \Lambda$, let $\mathcal{H}(F) = \{A \cap (X - F) : A \in \Omega(F)\}$ and let $\mathcal{R} = \cup_{F \in \Lambda} \mathcal{H}(F)$. Clearly each element $\mathcal{R}$ is a subset of some element of Ω. Also

$$\bigcup_{\mathcal{R}} V = \bigcup_{F \in \Lambda} \left[(X - F) \cap \bigcup_{\Omega(F)} A \right] = \bigcup_{F \in \Lambda} (X - F) = X$$

so $\mathcal{R}$ is an open refinement of Ω. We show that $\mathcal{R}$ is point–finite. If $x \in X$ then $x \in X - F$ for at most finitely many $F \in \Lambda$. Let Σ be the finite subset of Λ such that $F \in \Sigma$ implies $x \in X - F$. If $Q \in \mathcal{R}$ and $x \in Q$, it follows that $Q = A \cap (X - F) \in \mathcal{H}(F)$, where $F \in \Sigma$ and hence, x is an element of at most finitely many elements of $\mathcal{R}$. The proof is complete.

We state the following corollary to Theorem 1 without proof.

Corollary 1. A space is metacompact if and only if every closed filter of type $\mathcal{M}$ on the space has nonempty adherence.

We say that a space is countably compact if each sequence in the space has a cluster point (a point is a cluster point of a sequence if each open set about the point contains a subsequence of the sequence). It can readily be shown that a space is countably compact if and only if each filter with a countable base on the space has nonempty adherence. The characterization in Theorem 1 may be used to prove the following.

Theorem 2. A countably compact metacompact space X is compact.

Proof. Let Λ be a closed p.d. subcollection of a filter Ω on X. For $x \in X$, let $\Lambda(x) = \{F \in \Lambda : x \notin F\}$. Choose $x_0 \in X$ and for each positive integer n, choose

$x_n \in W_n = \cap_{k=0}^{n-1} \cap_{F \in \Lambda(x_k)} F$. Then $W_n \neq \emptyset$, W_n is closed and $W_{n+1} \subset W_n$. Since X is countably compact, choose $y \in X$ to be a cluster point of the sequence x_n. It follows that for all n, $y \in W_n \subset \cap_{k=0}^{\infty} \cap_{F \in \Lambda(x_k)} F$. Also, $y \in F \in \Lambda - \cup_{k=0}^{\infty} \Lambda(x_k)$ for each F, since $x_n \in F$ for each $F \in \Lambda - \cup_{k=0}^{\infty} \Lambda(x_k)$. Consequently adh $\Lambda \neq \emptyset$ and so Ω is of type $\mathcal{M}$. Thus, adh $\Omega \neq \emptyset$ since the space is metacompact. This completes the proof.

Lemma 2. If $g\colon X \to Y$ is continuous and Λ is a filter of type $\mathcal{M}$ on X, then the filter generated by $\{g(F) : F \in \Lambda\}$ is of type $\mathcal{M}$ on Y.

Proof. Let Γ be a closed p.d. subcollection of the filter generated by $\{g(F) : F \in \Lambda\}$. Then $\{g^{-1}(Q) : Q \in \Gamma\}$ is easily shown to be a closed p.d. subcollection of Λ. Thus, $\cap_{Q \in \Gamma} g^{-1}(Q) \neq \emptyset$. Since $g(\cap_{Q \in \Gamma} g^{-1}(Q)) \subset \cap_{Q \in \Gamma} Q$, we have adh $\Gamma \neq \emptyset$. This completes the proof.

Lemma 2 is especially interesting when we note that if Ω is an open covering of X, $\{g(V) : V \in \Omega\}$ need not be a collection of open sets.

Corollary 2. Let X be a space and $A \subset B$. Each filter of type $\mathcal{M}$ on A is a base for a filter of type $\mathcal{M}$ on B.

Proof. The identity function from the subspace A to the subspace B is continuous. An application of Theorem 2 completes the proof.

The following proofs of some known results exhibit how the result of Lemma 2 may be used. Hereafter, a base for a filter of type $\mathcal{M}$ will be referred to as a *filterbase of type* $\mathcal{M}$.

Theorem 3. A closed subspace of a metacompact space is metacompact.

Proof. By Corollary 2, a filterbase of type $\mathcal{M}$ on $A \subset X$ is a filterbase of type $\mathcal{M}$ on X and thus, has nonempty adherence in X. If A is closed then such a filterbase has nonempty adherence in A. The proof is complete.

Theorem 4. Each subspace of a metacompact space X is metacompact if and only if each open subspace is metacompact.

Proof. The necessity is obvious. Now suppose A is a subspace of X and that Ω is a filterbase of type $\mathcal{M}$ on A such that $A \cap \text{adh } \Omega = \emptyset$. Then $A \subset X - \text{adh } \Omega$, so Ω is a filterbase of type $\mathcal{M}$ on $X - \text{adh } \Omega$. This is a contradiction since $(X - \text{adh } \Omega) \cap \text{adh } \Omega = \emptyset$. The proof is finished.

Recall that a closed continuous $g\colon X \to Y$ is called perfect if $g^{-1}(v)$ is compact for each $v \in Y$.

Theorem 5. If $g: X \to Y$ is perfect and Y is metacompact, then X is metacompact.

Proof. Let Λ be a closed filterbase of type $\mathcal{M}$ on X. Then $\{g(F) : F \in \Lambda\}$ is a closed filterbase of type $\mathcal{M}$ on Y. Let $v \in \cap_{F\in\Lambda} g(F)$. Then $\{F \cap g^{-1}(v) : F \in \Lambda\}$ is a closed filterbase on $g^{-1}(v)$. Thus, $(\cap_{F\in\Lambda} F) \cap g^{-1}(v) \neq \emptyset$ so adh $\Lambda \neq \emptyset$ and the proof is complete.

Theorem 6. The product of a compact space and a metacompact space is metacompact.

Proof. The projection $p: X \times Y \to Y$ is perfect. The proof is complete.

An analysis of the proof of Theorem 2 leads us to a proof in terms of open sets. The proof given below should be compared with that given in [1]. For an open covering, Γ, of a space X, and $x \in X$, we will denote $\{V \in \Gamma : x \in V\}$ by $\Gamma(x)$. Evidently a space X is metacompact if and only if for every open covering of the space there is an open refinement Γ with $\Gamma(x)$ finite for each x. If λ is an infinite cardinal number, we will say that X is *metacompact of order* λ if for every open covering, there is an open refinement Γ with cardinality of $\Gamma(x) \leq \lambda$ for each x, and is λ–*compact* if every open covering of cardinality $\leq \lambda$ has a finite subcovering. A metacompact space is metacompact of order λ for each λ.

Theorem 7. A countably compact metacompact space X is compact.

Proof. It will be sufficient to show that each point–finite open covering of X contains a finite subcovering. Let Γ be a point–finite covering with no finite subcovering. Choose $x_0 \in X$ and for each natural number n choose $x_n \in W_n = X - \cup_{k=0}^{n-1} \cup_{V\in\Gamma(x_k)} V$. Then $W_n \neq \emptyset$ and W_n is closed. It is also easy to see that $W_{n+1} \subset W_n$. Let y be a cluster point of the sequence x_n. It is easy to show that $y \in \cap_{n=1}^{\infty} W_n = X - \cup_{k=0}^{\infty} \cup_{V\in\Gamma(x_k)} V$, and that $y \in X - V$ for $V \in \Gamma - \cup_{k=0}^{\infty} \Gamma(x_k)$. This contradiction completes the proof.

Theorem 8. If a space X is λ–compact and metacompact of order λ, the space is compact.

Proof. Using the same format as in the proof of Theorem 7, we note that $W_n \neq \emptyset$, since the cardinality of $\cup_{k=0}^{n-1} \Gamma(x_k) \leq \lambda$ and a λ–compact space is countably compact.

A space is said to have the Bolzano–Weierstrass Property if every infinite subset has a limit point. A countably compact space has the Bolzano–Weierstrass property and, if singletons are closed sets, a space with the Bolzano–Weierstrass property is countably compact. We see from the following example that a metacompact space which satisfies the Bolzano–Weierstrass property might fail to be compact. Let X be the set of positive integers and T be the topology on X with base $\{\{2n-1, 2n\} : n \in X\}$. Then (X, T) has the Bolzano–Weierstrass property, is metacompact, but is clearly not compact.

References

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James E. Joseph
Department of Mathematics
Howard University
Washington, D.C. 20059
email: jjoseph@fac.howard.edu

Myung H. Kwack
Department of Mathematics
Howard University
Washington, D.C. 20059
mkwack@howard.edu

Bhamini M. P. Nayar
Department of Mathematics
Morgan State University
Baltimore, MD 21251
bnayar@morgan.edu