THE MINIMAL RANK OF THE MATRIX EXPRESSION A – BX – YC

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Abstract. The minimal rank of the matrix expression $A - BX - YC$ with respect to the choice of X and Y are determined using generalized inverses of matrices. Some of their applications are also presented.

Suppose that

$$p(X, Y) = A - BX - YC \tag{1}$$

is a linear matrix expression over the complex number field, where A, B, and C are $m \times n$, $m \times k$, and $l \times n$ matrices, respectively; X and Y are $k \times n$ and $m \times l$ variant matrices, respectively. In this article we consider the minimal rank of $p(X, Y)$ with respect to the choice of X and Y, and present some of their applications. To do so, we need some well–known formulas related to ranks and generalized inverse of matrices.

Lemma 1 [2] [3]. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then they satisfy the rank equalities

$$r[A, B] = r(A) + r(B - AA^-B) = r(B) + r(A - BB^-A), \tag{2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^-A) = r(C) + r(A - AC^-C), \tag{3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[(I_m - BB^-)A(I_n - C^-C)], \tag{4}$$

where $(\cdot)^-$ denotes an inner inverse of a matrix.

We are ready to establish the main result of this article.

Theorem 2. The minimal rank of $p(X, Y)$ in (1) with respect to the choice of X and Y is

$$\min_{X,Y} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C). \tag{5}$$

The matrices X and Y satisfying (5) are given by

$$X = B^- A + UC + (I_k - B^- B)U_1, \tag{6}$$
$$Y = (I_m - BB^-)AC^- - BU + U_2(I_l - CC^-), \tag{7}$$

where U, U_1 and U_2 are arbitrary.

<u>Proof</u>. Let

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Then its rank obviously satisfies the inequality

$$r(M) \leq r(A) + r(B) + r(C). \tag{8}$$

Now replacing A in (8) by $p(X, Y)$ in (1), we obtain the following rank inequality

$$r \begin{bmatrix} A - BX - YC & B \\ C & 0 \end{bmatrix} \leq r(A - BX - YC) + r(B) + r(C). \tag{9}$$

It is easy to see by block elementary operations of matrices that

$$r \begin{bmatrix} A - BX - YC & B \\ C & 0 \end{bmatrix} = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Thus, (9) becomes

$$r(A - BX - YC) \geq r(M) - r(B) - r(C). \tag{10}$$

Observe that the right–hand side of (10) involves no X and Y. Thus, $r(M) - r(B) - r(C)$ is a lower bound for the rank of $p(X, Y)$ with respect to X and Y. On the other hand, putting (6) and (7) in $p(X, Y)$ yields

$$\begin{aligned} p(X, Y) &= A - BB^- A - BUC - (I_m - BB^-)AC^- C + BUC \\ &= (I_m - BB^-)A(I_n - C^- C). \end{aligned}$$

In that case, the rank of $p(X, Y)$ by (4) is

$$r[p(X, Y)] = r[(I_m - BB^-)A)(I_n - C^-C)] = r(M) - r(B) - r(C). \tag{11}$$

Combining (10) with (11), we know $r(M) - r(B) - r(C)$ is the minimal rank of $p(X, Y)$ with respect to X and Y, and the matrices of X and Y satisfying (5) are given by (6) and (7).

A direct consequence of Theorem 2 is given below, which was established in [1] and [8].

Corollary 3. Let $p(X, Y)$ be given by (1). Then the following statements are equivalent.
(a) $\min_{X,Y} r(A - BX - YC) = 0$.
(b) The matrix equation $BX + YC = A$ is solvable.
(c)

$$r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C).$$

(d) $(I_m - BB^-)A(I_n - C^-C) = 0$.
In that case, the general solution of $BX + YC = A$ is

$$X = B^-A + UC + (I_k - B^-B)U_1, \tag{12}$$
$$Y = (I_m - BB^-)AC^- - BU + U_2(I_l - CC^-). \tag{13}$$

Observe that (12) and (13) have the same form as (1). Thus, we can also find the minimal ranks of solutions of $BX + YC = A$ when it is solvable.

Corollary 4. Suppose that the matrix equation $BX + YC = A$ is solvable. Then the minimal ranks of solutions X and Y to $BX + YC = A$ are

$$\min_{BX+YC=A} r(X) = r \begin{bmatrix} A \\ C \end{bmatrix} - r(C), \tag{14}$$

and

$$\min_{BX+YC=A} r(Y) = r[A, B] - r(B). \tag{15}$$

Proof. Since $BX + YC = A$ is solvable, it follows by Corollary 3(d) that

$$A - BB^- A - AC^- C + BB^- AC^- C = 0.$$

In that case, applying (5) and then (3) to (12) produces

$$\begin{aligned}
\min_{BX+YC=A} r(X) &= \min_{U, U_1} r[B^- A + UC + (I_k - B^- B)U_1] \\
&= r\begin{bmatrix} B^- A & I_k - B^- B \\ C & 0 \end{bmatrix} - r(I_k - B^- B) - r(C) \\
&= r\begin{bmatrix} B^- A & I_k \\ C & 0 \\ 0 & B \end{bmatrix} - r(B) - r(I_k - B^- B) - r(C) \\
&= r\begin{bmatrix} 0 & I_k \\ C & 0 \\ BB^- A & 0 \end{bmatrix} - k - r(C) \\
&= r\begin{bmatrix} C \\ BB^- A \end{bmatrix} - r(C) \\
&= r\begin{bmatrix} C \\ A - AC^- C + BB^- AC^- C \end{bmatrix} - r(C) = r\begin{bmatrix} C \\ A \end{bmatrix} - r(C),
\end{aligned}$$

establishing (14). Similarly, we can derive (15) from (13) and (5).

Theorem 5. Suppose that the two linear matrix equations

$$A_1 X_1 B_1 = C_1 \quad \text{and} \quad A_2 X_2 B_2 = C_2 \tag{16}$$

are solvable, respectively, where X_1 and X_2 are $k \times l$ matrices. Then
(a) The minimal rank of the difference $X_1 - X_2$ of two solutions of (16) is

$$\min_{\substack{A_1 X_1 B_1 = C_1 \\ A_2 X_2 B_2 = C_2}} r(X_1 - X_2) = r\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - r[B_1, B_2]. \tag{17}$$

(b) [5] [6] In particular, the pair of matrix equations in (16) have a common solution if and only if

$$r\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r[B_1, B_2]. \tag{18}$$

Proof. It is well–known (see [7]) that a matrix equation of the form $AXB = C$ is solvable if and only if $AA^-C = C$ and $CB^-B = C$ hold. In that case, the general solution $AXB = C$ can be written as

$$X = A^-CB^- + (I_k - A^-A)U + V(I_l - BB^-),$$

where U and V are arbitrary. If the two equations in (16) are solvable, respectively, their general solutions can be written as

$$X_1 = A_1^-C_1B_1^- + (I_k - A_1^-A_1)U_1 + V_1(I_l - B_1B_1^-),$$

and

$$X_2 = A_2^-C_2B_2^- + (I_k - A_2^-A_2)U_2 + V_2(I_l - B_2B_2^-),$$

where U_1, V_1, U_2 and V_2 are arbitrary. In that case,

$$X_1 - X_2 =$$

$$A_1^-C_1B_1^- - A_2^-C_2B_2^- + [I_k - A_1^-A_1, I_k - A_2^-A_2]\begin{bmatrix} U_1 \\ -U_2 \end{bmatrix} + [V_1, -V_2]\begin{bmatrix} I_l - B_1B_1^- \\ I_l - B_2B_2^- \end{bmatrix}.$$

Thus, by (5) we find that

$$\min_{\substack{A_1X_1B_1=C_1 \\ A_2X_2B_2=C_2}} r(X_1 - X_2) = r\begin{bmatrix} A_1^-C_1B_1^- - A_2^-C_2B_2^- & I_k - A_1^-A_1 & I_k - A_2^-A_2 \\ I_l - B_1B_1^- & 0 & 0 \\ I_l - B_2B_2^- & 0 & 0 \end{bmatrix}$$

$$- r\begin{bmatrix} I_l - B_1B_1^- \\ I_l - B_2B_2^- \end{bmatrix} - r[I_k - A_1^-A_1, I_k - A_2^-A_2]. \tag{19}$$

Simplifying by (2) and (3) the ranks of the above three block matrices, we get

$$
\begin{aligned}
& r\begin{bmatrix} A_1^- C_1 B_1^- - A_2^- C_2 B_2^- & I_k - A_1^- A_1 & I_k - A_2^- A_2 \\ I_l - B_1 B_1^- & 0 & 0 \\ I_l - B_2 B_2^- & 0 & 0 \end{bmatrix} \\
& = r\begin{bmatrix} A_1^- C_1 B_1^- - A_2^- C_2 B_2^- & I_k & I_k & 0 & 0 \\ I_l & 0 & 0 & B_1 & 0 \\ I_l & 0 & 0 & 0 & B_2 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{bmatrix} \\
& \quad - r(A_1) - r(A_2) - r(B_1) - r(B_2) \\
& = r\begin{bmatrix} 0 & I_k & 0 & 0 & 0 \\ I_l & 0 & 0 & B_1 & 0 \\ I_l & 0 & 0 & 0 & B_2 \\ -C_1 B_1^- & 0 & -A_1 & 0 & 0 \\ C_2 B_2^- & 0 & A_2 & 0 & 0 \end{bmatrix} - r(A_1) - r(A_2) - r(B_1) - r(B_2) \\
& = r\begin{bmatrix} 0 & I_k & 0 & 0 & 0 \\ I_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_1 & B_2 \\ 0 & 0 & -A_1 & C_1 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 \end{bmatrix} - r(A_1) - r(A_2) - r(B_1) - r(B_2) \\
& = r\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} + k + l - r(A_1) - r(A_2) - r(B_1) - r(B_2),
\end{aligned}
$$

$$
r\begin{bmatrix} I_l - B_1 B_1^- \\ I_l - B_2 B_2^- \end{bmatrix} = r\begin{bmatrix} I_l & B_1 & 0 \\ I_l & 0 & B_2 \end{bmatrix} - r(B_1) - r(B_2) = r[B_1, B_2] + l - r(B_1) - r(B_2),
$$

and

$$r[I_k - A_1^- A_1, I_k - A_2^- A_2] = r\begin{bmatrix} I_k & I_k \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} - r(A_1) - r(A_2)$$

$$= r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + k - r(A_1) - r(A_2).$$

Putting the above three in (19) yields (17). The result in part (b) is an immediate consequence of (17).

Corollary 6. Let A and B be two matrices of the same size. Then

(a) The minimal rank of the difference of $A^- - B^-$ of two inner inverses of A and B is

$$\min_{A^-, B^-} r(A^- - B^-) = r(A - B) + r(A) + r(B) - r[A, B] - r\begin{bmatrix} A \\ B \end{bmatrix}. \tag{20}$$

(b) In particular, A and B have a common inner inverse if and only if

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B). \tag{21}$$

Proof. Notice that A^- and B^- are solutions of the matrix equations $AXA = A$ and $BYB = B$, respectively. Thus (20) follows from (17).

Corollary 7. Let A and B be any two idempotent matrices of the same size. Then their difference $A - B$ satisfies the two rank equalities

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B), \tag{21}$$

and

$$r(A - B) = r(A - AB) + r(AB - B). \tag{22}$$

Proof. Notice that any two idempotent matrices of the same size have the identity matrix as their common inner inverse. Thus (21) follows immediately from Corollary 6(b). When A and B are idempotent, we also find by (2) and (3) that

$$r\begin{bmatrix} A \\ B \end{bmatrix} = r(B) + r(A - AB) \quad \text{and} \quad r[A, B] = r(A) + r(B - AB).$$

Putting them in (21) yields (22).

Corollary 8. Let A be a given matrix, and let X and Y be any two outer inverses of A, that is, $XAX = X$ and $YAY = Y$. Then their difference of $X - Y$ satisfies the rank equality

$$r(X - Y) = r\begin{bmatrix} X \\ Y \end{bmatrix} + r[X, Y] - r(X) - r(Y). \tag{23}$$

Proof. Obviously, any two outer inverses of the matrix A have A as their common inner inverse. Thus (23) follows immediately from Corollary 6(b).

On the basis of Corollaries 7 and 8, one can derive a variety of results related to idempotent matrices and outer inverses of matrices. We shall present them in other papers.

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