# ON SECOND ORDER INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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**Abstract.** In this paper we investigate the existence of mild solutions on the semi–infinite interval to second order initial value problems for a class of integrodifferential inclusions in Banach spaces. We shall make use of a theorem of Ma, which is an extension to multivalued maps on locally convex topological spaces of Schaefer's Theorem.

**1. Introduction.** In the past few years several papers have been devoted to the study of the existence of mild solutions for differential equations in abstract spaces. We refer to the books of Barbu [3], Goldstein [9], Heikkila and Lakshmikantham [10], Ladas and Lakshmikantham [13], Lakshmikantham and Leela [14], and Zaidman [24], and to the papers of Fattorini [6, 7], Heikkila and Lakshmikantham [11], Kusano and Oharu [12], Lakshmikantham and Leela [14], Martin [18] and Travis and Webb [21, 22]. However, very few results are available for evolution inclusions on compact intervals, see Avgerinov and Papageorgiou [2] and Papageorgiou [19].

In [4] Benchohra gives an existence theorem for mild solutions on unbounded intervals for a class of first order multivalued problems.

In this paper we shall prove a theorem which assures the existence of mild solutions defined on an unbounded real interval $J$ for the initial value problem (IVP for short) of the second order integrodifferential inclusion

$$y'' - Ay \in \int_0^t K(t,s)F(s,y)\,ds, \quad t \in J, \quad y(0) = y_0, \quad y'(0) = y_1 \tag{1.1}$$

where $F\colon J \times E \to E$ is a bounded, closed, convex valued multivalued map, $K\colon D \to \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, $y_0, y_1 \in E$, $J$ an unbounded real interval, $A$ a linear operator from a dense subspace $D(A)$ of $E$ into $E$ and $E$ a real Banach space with norm $|\cdot|$. For the sake of simplicity, we choose $J = [0, +\infty)$.

The method we are going to use is to reduce the existence of solutions to (1.1) to the search for fixed points of a suitable multivalued map on the Fréchet space $C(J,E)$. In order to prove the existence of fixed points, we shall rely on a theorem due to Ma [17], which is an extension to multivalued maps between locally convex topological spaces of Schaefer's Theorem [20].

**2. Preliminaries.** In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. $J_m$ is the compact real interval $[0, m]$ $(m \in \mathbb{N})$. $C(J, E)$ is the linear metric Fréchet space of continuous functions from $J$ into $E$ with the following metric [8]

$$d(y, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y - z\|_m}{1 + \|y - z\|_m} \quad \text{for each } y, z \in C(J, E),$$

where

$$\|y\|_m := \sup\{|y(t)| : t \in J_m\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from $E$ into $E$.
A measurable function $y\colon J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral we refer to Yosida [23].
$L^1(J, E)$ denotes the Banach space of continuous functions $y\colon J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{\infty} |y(t)|\, dt \quad \text{for all} \quad y \in L^1(J, E).$$

$U_p$ denotes the neighborhood of 0 in $C(J, E)$ defined by

$$U_p := \{y \in C(J, E) : \|y\|_m \leq p \text{ for each } m \in \mathbb{N}\}.$$

The convergence in $C(J, E)$ is the uniform convergence on compact intervals, i.e. $y_j \to y$ in $C(J, E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \to 0$ in $C(J_m, E)$ as $j \to \infty$. $M \subseteq C(J, E)$ is a bounded set if and only if there exists a positive function $\phi \in C(J, \mathbb{R}_+)$ such that

$$|y(t)| \leq \phi(t) \quad \text{for all} \quad t \in J \quad \text{and all} \quad y \in M.$$

The Ascoli–Arzela theorem says that a set $M \subseteq C(J, E)$ is compact if and only if for each $m \in \mathbb{N}$, $M$ is a compact set in the Banach space $(C(J_m, E), \|\cdot\|_m)$.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ ($I$ is the identity operator in $E$),
(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in R$,
(iii) The map $t \mapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A: E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(0)y.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [9] and to the papers of Fattorini [6, 7] and Travis and Webb [21, 22].

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G: X \to X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. $G$ is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in $X$ for any bounded set $B$ of $X$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

$G$ is called upper semicontinuous (u.s.c.) on $X$ if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of $X$, and if for each open set $B$ of $X$ containing $G(x_0)$, there exists an open neighborhood $A$ of $x_0$ such that $G(A) \subseteq B$.

$G$ is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph (i.e. $x_n \to x_0$, $y_n \to y_0$, $y_n \in Gx_n$ imply $y_0 \in Gx_0$). $G$ has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of $X$.

A multivalued map $G: J \to BCC(E)$ is said to be measurable if for each $x \in E$ the distance between $x$ and $G(t)$ is a measurable function on $J$. For more details on multivalued maps see Aubin and Frankowska [1] and Deimling [5].

Let us list the following hypotheses:

(H1) $A$ is an infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$;

(H2) $F: J \times E \to BCC(E)$; $(t, y) \mapsto F(t, y)$ is strongly measurable with respect to $t$ for each $y \to E$, u.s.c. with respect to $y$ for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \{g \in L^1(J, E) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}$$

is nonempty;

(H3) For each $t \in J_m$ $(m \in \{1, 2, \dots\})$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess } \sup\{|K(t, s)| \ , \ 0 \leq s \leq t\},$$

is bounded on $J_m$;

(H4) The map $t \mapsto K_t$ is continuous from $J_m$ to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H5) $\|F(t, y)\| := \sup\{|v| \in F(t, y)\} \leq p(t)\psi(|y|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi: \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with

$$Mm \sup_{t \in J_m} K(t) \int_0^m p(s)\, ds < \int_0^\infty \frac{du}{\psi(u)} \quad \text{for each } m \in \mathbb{N};$$

where $c = M|y_0| + Mm|y_1|$ and $M = \sup\{\|C(t)\|; t \in J\}$;

(H6) For each neighborhood $U_p$ of 0, $y \in U_p$ and $t \in J$ the set

$$\left\{ C(t)y_0 + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s, u)g(u)\, du\, ds \ : \ g \in S_{F,y} \right\}$$

is relatively compact.

Remark 2.1. If $\dim E < \infty$ and $J$ is a compact real interval, then for each $y \in C(J, E)$ $S_{F,y} \neq \emptyset$ (see Lasota and Opial [16]).

Definition 2.1. A continuous solution $y(t)$ of the integral inclusion

$$y(t) \in C(t)y_0 + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s, u)F(u, y(u))\, du\, ds$$

is called a mild solution on $J$ of (1.1).

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.1. [16] Let $I$ be a compact real interval and $X$ be a Banach space. Let $F$ be a multivalued map satisfying (H2) and let $\Gamma$ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$, then the operator

$$\Gamma \circ S_F : C(I, X) \to BCC(C(I, X)), \ y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X)\times C(I,X)$.

Lemma 2.2. [17] Let $X$ be a locally convex space and $N\colon X\to X$ be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighborhood $U_p$ of 0 for which $N(U_p)$, $p\in\mathbb{N}$ is a relatively compact set. If the set

$$\Omega := \{y\in X : \lambda y\in N(y) \ \text{ for some } \lambda>1\}$$

is bounded, then $N$ has a fixed point.

**3. Main Result.** Now, we are able to state and prove our main theorem.

Theorem 3.1. Assume that (H1) through (H6) hold. Then the IVP (1.1) has at least one mild solution on $J$.

Proof. We transform the problem into a fixed point problem. Consider the multivalued map, $N\colon C(J,E)\to C(J,E)$ defined by

$$Ny := \Big\{h\in C(J,E) : h(t) = C(t)y_0 + S(t)y_1$$

$$+\int_0^t S(t-s)\int_0^s K(s,u)g(u)\,du\,ds : g\in S_{F,y}\Big\}$$

where

$$S_{F,y} = \Big\{g\in L^1(J,E) : g(t)\in F(t,y(t)) \ \text{ for a.e. } t\in J\Big\}.$$

It is clear that the fixed points of $N$ are mild solutions to (1.1). We shall show that $N$ satisfies the assumptions of Lemma 2.2. The proof will be given in three steps.

Step 1. Clearly, (H2) shows that $Ny$ is convex for each $y\in C(J,E)$. By (H3) through (H5), $N(U_q)$ is bounded and equicontinuous for each $q\in\mathbb{N}$.

As a consequence of Step 1 and (H6) together with the Ascoli–Arzela theorem we can conclude that $N(U_p)$ is relatively compact in $C(J,E)$.

Step 2. $N$ has a closed graph. Let $y_n\to y_*$, $h_n\in Ny_n$, and $h_n\to h_0$. We shall prove that $h_0\in Ny_*$. $h_n\in N(y_n)$ means that there exists $g_n\in S_{F,y_n}$ such that

$$h_t(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g_n(u)\,du\,ds.$$

We must prove that there exists $g_0 \in S_{F,y_*}$ such that

$$h_0(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g_0(u)\,du\,ds. \qquad (3.1)$$

The idea is then to use the facts that
(i) $h_n \to h_0$;
(ii) $h_n - C(t)y_0 - S(t)y_1 \in \Gamma(S_{F,y_n})$ where

$$\Gamma\colon L^1(J,E) \to C(J,E) \quad \text{defined by } (\Gamma g)(t) := \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,du\,ds.$$

If $\Gamma \circ S_F$ is a closed graph operator, we would be done. But we do not know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cut the functions $y_n$, $h_n - C(t)y_0 - S(t)y_1$, $g_n$ and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbb{N}$. Then, using Lemma 2.1, in this case we are able to affirm that (3.1) is true on the compact interval $[k, k+1]$, i.e.

$$h_0(t)\Big|_{[k,k+1]} = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g_0^k(u)\,du\,ds$$

for suitable $L^1$–selection $g_0^k$ of $F(t, y_*(t))$ on the interval $[k, k+1]$. At this point we can paste the functions $g_0^k$ obtaining the selection $g_0$ defined by

$$g_0(t) = g_0^k(t) \quad \text{for } t \in [k, k+1).$$

We obtain then that $g_0$ is an $L^1$–selection and (3.1) will be satisfied. We now give the details.
Clearly we have that

$$\|(h_n - C(t)y_0 - S(t)y_1) - (h_0 - C(t)y_0 - S(t)y_1\|_\infty \to 0, \quad \text{as } n \to \infty.$$

Now, we consider for all $k \in \mathbb{N}$, the mapping

$$S_F^k\colon C([k, k+1], E) \to L^1([k, k+1], E)$$

$$u \mapsto S_{F,u}^k := \{f \in L^1([k, k+1], E) : f(t) \in F(t, u(t)) \ \text{ for a.e. } t \in [k, k+1]\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k\colon L^1([k,k+1],E) \to C([k,k+1],E)$$

$$g \mapsto \Gamma_k(g)(t) = \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,du\,ds.$$

From Lemma 2.1, it follows that $\Gamma_k \circ S_F^k$ is a closed graph operator for all $k \in \mathbb{N}$. Moreover, we have that

$$\big(h_n(t) - C(t)y_0 - S(t)y_1\big)\Big|_{[k,k+1]} \in \Gamma_k(S^k_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.1 that

$$\big(h_0(t) - C(t)y_0 - S(t)y_1\big)\Big|_{[k,k+1]} = \int_0^t S(t-s)\int_0^s K(s,u)g_0^k\,du\,ds$$

for some $g_0^k \in S^k_{F,y_*}$. So the function $g_0$ defined on $J$ by

$$g_0(t) = g_0^k(t) \quad \text{for } t \in [k,k+1)$$

is in $S_{F,y_*}$ since $g_0(t) \in F(t,y_*(t))$ for a.e. $t \in J$.

Step 3. Now it remains to show that the set $\Omega$ is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus, there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}C(t)y_0 + \lambda^{-1}S(t)y_1 + \lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,u)g(u)\,du\,ds, \quad t \in J.$$

This implies by (H3) through (H5) that for each $t \in J_m$ we have

$$
\begin{aligned}
|y(t)| &\le M|y_0| + Mm|y_1| + M\left\| \int_0^t \int_0^s K(s,u)g(u)\,du\,ds \right\| \\
&\le M|y_0| + Mm|y_1| + M\int_0^t \int_0^s |K(s,u)|p(u)\psi(|y(u)|)\,du\,ds \\
&\le M|y_0| Mm|y_1| + Mm \sup_{t\in J_m} K(t) \int_0^t p(s)\psi(|y(s)|)\,ds.
\end{aligned}
$$

Let us take the right–hand side of the above inequality as $v(t)$, then we have

$$v(0) = M|y_0| + Mm|y_1| \ \text{ and } \ |y(t)| \le v(t).$$

Using the increasing character of $\psi$ we get

$$v'(t) \le Mm \sup_{t\in J_m} K(t)p(t)\psi(v(t)).$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le Mm \sup_{t\in J_m} K(t) \int_0^m p(s)\,ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant $b$ such that $v(t) \le b$, $t \in J_m$, and hence, $\|y\|_\infty \le b$ where $b$ depends on $m$ and on the functions $p$ and $\psi$. This shows that $\Omega$ is bounded. Set $X := C(J,E)$. As a consequence of Lemma 2.2 we deduce that $N$ has a fixed point which is a mild solution of (1.1).

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