# MODERN MATHEMATICAL MILESTONES: MORLEY'S MYSTERY

Richard L. Francis

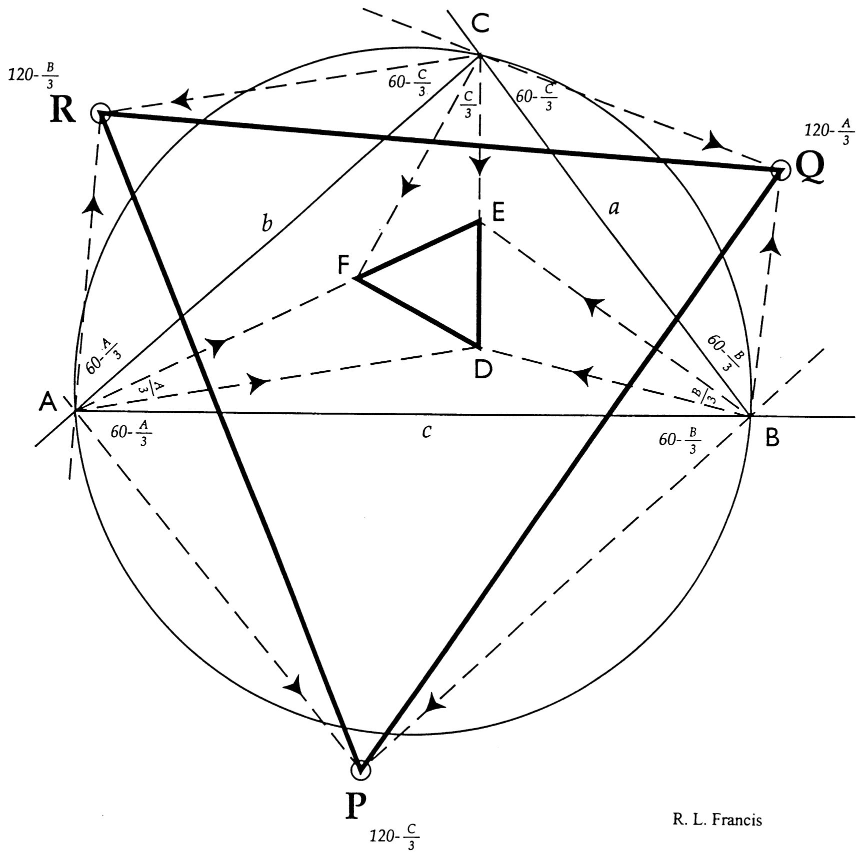
A Euclidean result, discovered only in modern times, is the Morley Triangle Theorem. It reveals that the corresponding angle trisectors of any triangle intersect in the vertices of an equilateral triangle. This resulting triangle, named after Frank Morley (1860–1937), is to trisectors what the incenter of a triangle is to angle bisectors. Apparently overlooked by ancient geometers or hastily abandoned because of trisection and constructibility uncertainties, the problem came to light only a century ago. Though conjectured around 1900 by Frank Morley, resolution or rigorous proof was to await even more recent advancements. This beautiful and elegant Euclidean theorem, mysteriously unnoticed across the ages, thus belongs to the twentieth century.

One of the first to prove the Morley Triangle Theorem was M. T. Naraniengar in or around 1909. Today, proofs of many kinds are known; some are direct, some indirect. Both analytic and trigonometric proofs supplement the elephantine proofs of a purely geometric kind.

The problem continues to shed its abundant mysteries. Among these is the case for exterior angle trisectors as opposed to interior trisectors. The following figure, prepared for the writer by the university's graphics department, strongly suggests that corresponding exterior angle trisectors of any triangle also intersect in the vertices of an equilateral triangle. Personal proofs, first for special cases, and then in general by *MATHEMATICA,* rest on angle measure labels of the accompanying figure. An earlier proof is attributed to G. L. Niedhardt and V. Milenkovic in the late 1960s. The result was suggested in large measure by its bisector counterpart in the determination of the excenters of the general triangle.

Various questions, here unanswered, command our attention. Note that $Z$ is the circumradius of the given triangle. Whereas side $s$ (say $DE$) of the interior Morley triangle of triangle $ABC$ is given by

$$s = 8Z \sin\frac{A}{3} \sin\frac{B}{3} \sin\frac{C}{3},$$




# THE EXTENDED MORLEY CONFIGURATION

does it also follow that side $S$ (say $PQ$) of the exterior Morley triangle yields the formula

$$S = 8Z \sin \frac{\pi - A}{3} \sin \frac{\pi - B}{3} \sin \frac{\pi - C}{3}?$$

Of course, the processes can be repeated so as to consider Morley triangles of Morley triangles. Interior angle trisectors successively form an infinite regression of such triangles whereas exterior trisectors form an infinite progression. Thus, to what point are the inner triangles converging?

All triangles in the regressive and progressive pattern are equilateral though the reference triangle $ABC$ may or may not be. As angle trisection is generally impossible by use of the Euclidean tools, the only constructible triangles in the overall regressive and progressive setting can occur at the outset. Examples such as the right isosceles triangle bear this out. As the 60 degree angle cannot be trisected with the Euclidean instruments, the Morley triangle of a Morley triangle is thus not constructible. Of course, trisectors exist whether constructible or not.

The mystery continues as consideration is given to various perimeter and area comparisons in the infinite set of triangles, or to select lines and resulting angle considerations (lines such as $\overleftrightarrow{QE}$ or $\overleftrightarrow{RF}$ or $\overleftrightarrow{PD}$), or to non-Euclidean geometry.

The Morley Triangle Theorem provides a fertile field of study for the modern day geometer in the unraveling of its mysteries. This is so in spite of overtones of antiquity. Interestingly, it is reminiscent in many ways of the famous Lehmus-Terquem-Steiner Theorem for angle bisectors as discovered only in the nineteenth century. Such a mathematical state of belated discovery raises an intriguing question as to "what remarkable, yet unnoticed, Euclidean theorem awaits as the twenty-first century unfolds?"

## *References*

1. R. L. Francis, "A Renaissance of Geometric Constructions," *Missouri Journal of Mathematical Sciences,* 8 (1996), 113–124.

2. R. L. Francis, "Cubic Connections: An Attack on Antiquity, *Pi Mu Epsilon Journal,* 10 (1997), 528–531.

3. R. L. Francis, "The Morley Triangle Problem," *Consortium,* 66 (1998), 8–9.

Richard L. Francis
Department of Mathematics
Southeast Missouri State University
Cape Girardeau, MO 63701
email: c714scm@semovm.semo.edu