HOMOTOPIC CLASSIFICATION OF EULER–LAGRANGE SYSTEMS

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Abstract. In this paper we examine the linear elliptic partial differential operators that appear as Euler–Lagrange systems of certain variational integrals. We give a sufficient condition for those systems to be deformable to the Laplace system.

1. Introduction. Let Ω denote a bounded domain in $\mathbb{R}^2$. We shall consider linear elliptic partial differential operators acting on functions of Sobolev class $W_2^1(\Omega)$, the space of L^2–functions whose first–order derivatives are L^2–integrable. We focus our attention in particular on such operators that arise as the Euler–Lagrange systems of certain variational integrals.

Consider the functional:

$$I(u,v) = \iint_\Omega P(\nabla u, \nabla v)\,dxdy \tag{1.1}$$

where $P(X,Y)$ is a homogeneous quadratic polynomial with respect to (X,Y) in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. This functional is well–defined for $u, v \in W_2^1(\Omega)$.

A necessary condition for (u,v) to be a stationary point for the functional I is the following:

$$\begin{cases} \operatorname{div} P_X(\nabla u, \nabla v) = 0 \\ \operatorname{div} P_Y(\nabla u, \nabla v) = 0. \end{cases} \tag{1.2}$$

Here, $P_X\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $P_Y\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the gradients of $P(X,Y)$ with respect to X and Y, respectively. System (1.2) is known as the Euler–Lagrange system for the functional I and is understood in the sense of distribution.

In this paper we describe those integrands P for which the Euler–Lagrange system (1.2) is deformable to the Laplace system

$$\begin{cases} \operatorname{div} \nabla u = 0 \\ \operatorname{div} \nabla v = 0. \end{cases}$$

Our main theorem is the following.

Theorem 1. Suppose that $P(X,Y) \neq 0$ for every $X \in \mathbb{R}^2 - \{0\}$ and every $Y \in \mathbb{R}^2 - \{0\}$. Then (1.2) is continuously deformable to the Laplace system within the class of elliptic systems with constant coefficients.

Next we look at an example where $P(X,Y)$ changes signs. We show that even if $P(X,Y)$ is not rank–one convex, the corresponding Euler–Lagrange system is still elliptic but now deformable to a system that we will call the *Bitsadze* system.

The converse of Theorem 1 is also investigated. We give an example to show that, unless some conditions on $P(X,Y)$ are satisfied, the converse does not hold.

2. Homotopic Classification of Elliptic Operators. Let $Q: \mathbb{R}^2 \to GL(2)$ denote a homogeneous polynomial of degree 2 with values in $GL(2)$, the general linear group on $\mathbb{R}^2$; in other words,

$$Q(\xi) = \sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j, \text{ for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{2.1}$$

Here, A_{ij} are 2×2 matrices with real entries such that $A_{ij} = A_{ji}$ for $i,j = 1,2$.

Q induces a second–order linear differential operator $Q(D)$ acting on vector functions $U = (u,v)$ by the following rule.

$$Q(D)U = \sum_{i,j=1}^{2} A_{ij}\frac{\partial^2 U}{\partial x_i \partial x_j}. \tag{2.2}$$

The *characteristic polynomial* of $Q(D)$ is then defined by

$$p(\xi) = \det\Big(\sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j\Big). \tag{2.3}$$

Definition 2.1. The operator $Q(D)$ and the system $Q(D)U = 0$ are said to be *elliptic* if $p(\xi) \neq 0$ for $\xi \in S^1$.

In what follows we shall also look at system (2.2) as a single equation of a complex function $u + iv$ in one complex variable $z = x + iy$.

Let $\mathbb{P}$ denote the set of all operators of the form (2.2). Observe that each operator Q in $\mathbb{P}$ is determined by 12 real numbers, the entries of the 2×2 matrices $A_{11}, A_{12} = A_{21}, A_{22}$. Therefore we may identify $\mathbb{P}$ with the Euclidean space of dimension 12.

Let $\mathbb{E}$ denote the subset of $\mathbb{P}$ consisting only of elliptic operators. It follows from the compactness of S^1 that a small perturbation of the matrices A_{ij} does not destroy the ellipticity of the operators $Q(D)$. This means exactly that the set $\mathbb{E}$ of elliptic operators is an open subset of $\mathbb{P}$. The components of the set $\mathbb{E}$ can be used to classify elliptic operators. The idea is to find representatives of each component so that any elliptic operator can be deformed continuously to exactly one of these representatives. This is to say that there is a homotopy connecting that operator with the corresponding representative.

The homotopic classification of second–order elliptic operators is due to B. Bojarski [1, 2, 3, 6]. Using complex rotation, Bojarski [3] showed that $\mathbb{E}$ has exactly six components represented by

$$\frac{\partial^2}{\partial z \partial z}, \ \frac{\partial^2}{\partial \overline{z} \partial \overline{z}}, \ \frac{\partial^2}{\partial z \partial \overline{z}}, \ \frac{\overline{\partial^2}}{\partial z \partial z}, \ \frac{\overline{\partial^2}}{\partial \overline{z} \partial \overline{z}}, \ \frac{\overline{\partial^2}}{\partial z \partial \overline{z}}.$$

Here, $z = x + iy$ and $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}}$ are the Cauchy–Riemann operators given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Observe that $4\frac{\partial^2}{\partial z \partial \overline{z}}$ is the complex form of the familiar Laplace operator.

3. Euler–Lagrange Systems. As was seen in the introduction, some second–order systems arise as variational equations, see (1.2), of a functional of the form (1.1). In general such systems are not elliptic. We examine under what conditions on $P(X, Y)$ systems (1.2) are elliptic and give their homotopic classification. The emphasis will be on those systems that are deformable to the Laplacian.

It is a well–known fact from linear algebra that the polynomial $P(X,Y)$ can be expressed as

$$P(X,Y) = \langle AX, X\rangle + 2\langle BX, Y\rangle + \langle CY, Y\rangle, \tag{3.1}$$

where A and C are 2×2 symmetric matrices, B is any 2×2 matrix and $\langle\cdot,\cdot\rangle$ stands for the usual inner product in $\mathbb{R}^2$.

The gradients of $P(X,Y)$ with respect to X and Y are

$$\begin{cases} P_X = 2AX + 2B^tY \\ P_Y = 2BX + 2CY. \end{cases}$$

Here, the symbol B^t denotes the transpose of the matrix B.

System (1.2) then becomes

$$\begin{cases} \operatorname{div}(A\nabla u + B^t\nabla v) = 0 \\ \operatorname{div}(B\nabla u + C\nabla v) = 0. \end{cases} \tag{3.2}$$

Now we apply the elementary identity

$$\operatorname{div}(B\nabla w) = \operatorname{div}(B^t\nabla w)$$

which holds for any function w. Therefore, system (3.2) will not change if we replace B by its symmetric part $\frac{B+B^t}{2}$. For this reason we may assume that B is a symmetric matrix.

Remark. Recall that the integrand of a variational function is said to be null–Lagrangian or quasi–affine if every functional is a stationary point for the corresponding Euler–Lagrange system. For instance, the integrand $\langle D\nabla u, \nabla v\rangle$ is a null–Lagrangian if and only if D is an anti–symmetric matrix. This is a general fact that we have actually used when we assumed that B is symmetric. For the interested reader we refer to more general null–Lagrangians introduced and examined by J. Ball, J. Currie and P. Oliver [4].

It is straightforward to verify that the characteristic polynomial for system (3.2) is

$$p(\xi) = \det\begin{bmatrix} \langle A\xi,\xi\rangle & \langle B\xi,\xi\rangle \\ \langle B\xi,\xi\rangle & \langle C\xi,\xi\rangle \end{bmatrix}.$$

The proposition below now follows immediately.

Proposition 3.1. The Euler–Lagrange system for the functional I as given in (1.1) with quadratic integrand (3.1) is elliptic if and only if

$$p(\xi) = \langle A\xi, \xi\rangle\langle C\xi, \xi\rangle - \langle B\xi, \xi\rangle^2 \neq 0,\ \xi \in S^1.$$

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Suppose that $P(X, Y) > 0$ for every X and Y in $\mathbb{R}^2 - \{0\}$. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then the quadratic polynomial in λ, $P(X, \lambda Y) = \lambda^2\langle CY, Y\rangle + 2\lambda\langle BX, Y\rangle + \langle AX, X\rangle$, has negative discriminant. This means that

$$\langle AX, X\rangle\langle CY, Y\rangle > \langle BX, Y\rangle^2; X \neq 0,\ Y \neq 0. \tag{3.3}$$

If we let $X = Y = \xi \in S^1$, we obtain $p(\xi) > 0$, $\xi \neq 0$, and hence, by Proposition 3.1, system (3.2) is elliptic. Similarly, if $P(X, Y) < 0$, then again $P(\xi) > 0$ for all $\xi \neq 0$, so system (3.2) is elliptic.

What remains to show is that system (3.2) is deformable to the Laplace system. For this we first deform B to the zero matrix, then A and C to the identity matrix. In the process of these deformations, we shall not disrupt the ellipticity of the system. Since $P(\xi) > 0$, we see that $\langle A\xi, \xi\rangle\langle C\xi, \xi\rangle > \langle B\xi, \xi\rangle^2 \geq \langle (tB)\xi, \xi\rangle^2$, for $0 \leq t \leq 1$, $\xi \neq 0$. The ellipticity condition remains valid for all $0 \leq t \leq 1$. In particular, for $t = 0$ the system reduces to the case when $B = 0$. Next we deform A to the identity matrix with the aid of the homotopy $A_t = tA + (1 - t)I$, $0 \leq t \leq 1$. Notice that the characteristic polynomial

$$p_t(\xi) = \langle A_t\xi, \xi\rangle\langle C\xi, \xi\rangle = t\langle A\xi, \xi\rangle\langle C\xi, \xi\rangle + (1 - t)\langle \xi, \xi\rangle\langle C\xi, \xi\rangle$$

remains strictly positive for all $0 \leq t \leq 1$. Finally, we deform C to the identity by a similar homotopy, i.e., $C_t = tC + (1-t)I$, $0 \leq t \leq 1$. This completes the proof of Theorem 1.

Remark. The proof of Theorem 1 concluded that no matter what the sign of $P(X, Y)$ was (positive or negative), the characteristic polynomial $p(\xi)$ was shown to be always positive. There are however, elliptic systems with $p(\xi) < 0$.

These facts suggest that in order to obtain an Euler–Lagrange system for which $p(\xi) < 0$, we should consider polynomials $P(X, Y)$ which change signs.

Consider the variational integral

$$I(u,v) = \iint_\Omega [(u_x - v_y)^2 - (u_y + v_x)^2]dxdy. \tag{3.4}$$

Then, $P(X,Y) = P(x_1, x_2, y_1, y_2) = (x_1 - y_2)^2 - (x_2 + y_1)^2$ (which is not rank–one convex) can be either positive or negative. Here,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, $p(\xi) = (\xi_1^2 - \xi_2^2)(-\xi_1^2 + \xi_2^2) - (-2\xi_1\xi_2)^2 = -(\xi_1^2 + \xi_2^2) < 0$, and the Euler–Lagrange system is elliptic.

It is quite remarkable that the Euler–Lagrange system

$$\begin{aligned} u_{xx} - u_{yy} - 2v_{xy} &= 0 \\ v_{xx} - v_{yy} + 2u_{xy} &= 0 \end{aligned}$$

for functional (3.4) is the real variable notation for the *Bitsadze* system:

$$\frac{\partial^2 W}{\partial \overline{z} \partial \overline{z}} = 0.$$

In 1948, A. V. Bitsadze [5] showed that the boundary value problem

$$\begin{cases} \dfrac{\partial^2 W}{\partial \overline{z} \partial \overline{z}} = 0; & |z| < 1 \\ W(z) = 0; & |z| = 1 \end{cases}$$

is ill–posed. In fact, he showed that all functions of the form $(1 - |z|^2)f(z)$, where $f(z)$ is analytic on $|z| \leq 1$, are solutions to this problem. It is perhaps the sign of $p(\xi)$ that lies behind this phenomenon.

A natural question now arises. Does $p(\xi) > 0$ imply that $P(X,Y)$ has a constant sign? In general, this is incorrect unless $A = C$ and A is positive–definite.

Indeed, if $A = C$, then with the aid of the identity

$$\langle X, BY\rangle = \frac{1}{4}(\langle X+Y, B(X+Y)\rangle - \langle X-Y, B(X-Y)\rangle)$$

we obtain

$$4|\langle X, BY\rangle| < |\langle X+Y, B(X+Y)\rangle| + |\langle X-Y, B(X-Y)\rangle|,$$

or, since A is positive–definite,

$$4|\langle X, BY\rangle| < 2\langle AX, X\rangle + 2\langle AY, Y\rangle.$$

Let $\lambda \in \mathbb{R} - \{0\}$. Replacing Y by λY yields

$$4|\lambda|\ |\langle X, BY\rangle| < 2\langle AX, X\rangle + 2\lambda^2\langle AY, Y\rangle.$$

Letting $\lambda = \left(\frac{\langle AX,X\rangle}{\langle AY,Y\rangle}\right)^{\frac{1}{2}}$, we arrive at the inequality

$$\langle X, BY\rangle^2 < \langle AX, X\rangle\langle AY, Y\rangle,$$

as desired.

For $A \neq C$, we have a counter–example.

Counter Example. For $k \neq 0$, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k^2 \end{bmatrix},\ C = \begin{bmatrix} k^2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{k^2}{2} \\ \frac{k^2}{2} & 0 \end{bmatrix}.$$

Then $p(\xi) > 0$ while $P(X,Y)$ fails to have a constant sign. Indeed,

$$\langle AX, X\rangle = x_1^2 + k^2x_2^2,\ \langle CX, X\rangle = k^2x_1^2 + x_2^2.$$

Hence, $\langle X, BX\rangle^2 = k^4x_1^2x_2^2 < (1+k^4)x_1^2x_2^2 + k^2(x_1^4+x_2^4) = \langle AX, X\rangle\langle CX, X\rangle$, for $X = (x_1, x_2) \neq 0$. This shows that $p(\xi) > 0$ for $\xi \neq 0$. On the other hand, for $X = (1, 0)$ and $Y = (0, 1)$, a simple calculation shows that

$$\langle AX, X\rangle\langle CY, Y\rangle - \langle X, BY\rangle^2 = 1 - \frac{k^4}{4} < 0,\ k > 4^{\frac{1}{4}}.$$

In view of relation (3.3), this means that $P(X, Y)$ changes sign.

References

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