SOME BONNESEN–STYLE TRIANGLE INEQUALITIES

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Abstract. Some Bonnesen–style isoperimetric inequalities for triangles in the plane are presented. For example, it is shown that $L^2 - 12\sqrt{3}A \geq 35.098\ r(R-2r)$ for triangles with perimeter L, area A, inradius r, and circumradius R. Equality holds when and only when either the triangle is equilateral or the triangle is similar to the isosceles triangle with sides 1, 1, and λ where $\lambda \approx 1.23628634$ is the largest root of the equation $31x^3 - 28x^2 - 16x + 4 = 0$.

The classical isoperimetric inequality for convex sets says that amongst all convex sets in the plane with a given perimeter, the disc (i.e. a circle and its interior) is the one with the largest area. In symbols,

$$L^2 - 4\pi A \geq 0 \tag{1}$$

where L denotes the perimeter of the set and A denotes its area. The quantity, $L^2 - 4\pi A$ is known as the *isoperimetric deficit* for the set. It is of interest to find lower bounds (larger than 0) for the isoperimetric deficit.

During the 1920's, Bonnesen ([2], [3], [4]) found many such inequalities. These have come to be known as Bonnesen–style inequalities. For example,

$$L^2 - 4\pi A \geq (L - 2\pi r)^2 \tag{2}$$

$$L^2 - 4\pi A \geq (\frac{A}{r} - \pi r)^2 \tag{3}$$

$$L^2 - 4\pi A \geq \left(L - \frac{2A}{r}\right)^2 \tag{4}$$

$$L^2 - 4\pi A \geq (2\pi R - L)^2 \tag{5}$$

$$L^2 - 4\pi A \geq \pi^2(R - r)^2 \tag{6}$$

$$L^2 - 4\pi A \geq \left(\pi R - \frac{A}{R}\right)^2 \tag{7}$$

$$L^2 - 4\pi A \geq \left(L - \frac{2A}{R}\right)^2 \tag{8}$$

$$L^2 - 4\pi A \geq A^2\left(\frac{1}{r} - \frac{1}{R}\right)^2 \tag{9}$$

$$L^2 - 4\pi A \geq L^2\frac{(R-r)^2}{(R+r)^2} \tag{10}$$

where r denotes the radius of a circle inscribed in the set and R denotes the radius of the circle circumscribed about the set (see also [12] and [5]).

The analog for triangles of the isoperimetric inequality is well known [6]. Of all triangles with a given perimeter, the equilateral triangle is the one with the largest area. In other words, for triangles,

$$L^2 - 12\sqrt{3}A \geq 0. \tag{1'}$$

Furthermore, equality holds if and only if the triangle is equilateral.

In this note, we will find analogs for some of the Bonnesen–style inequalities for triangles. It is believed that the inequalities of Theorems 2, 3, and 4 are new since they do not appear in the standard reference works [6] and [10].

First note, as Osserman did in [12], that the following lemma is a direct consequence of simple algebraic manipulation.

Osserman's Lemma. If A, L, ρ, and π denote any positive real numbers, then the inequalities

$$L^2 - 4\pi A \geq (L - 2\pi\rho)^2$$

$$L^2 - 4\pi A \geq \left(\frac{A}{\rho} - \pi\rho\right)^2$$

$$L^2 - 4\pi A \geq \left(L - \frac{2A}{\rho}\right)^2$$

are each algebraically equivalent to

$$\rho L \geq A + \pi\rho^2.$$

We now state some Bonnesen–style inequalities for triangles.

Theorem 1. If A, L, r, and R denote the area, perimeter, inradius, and circumradius of a triangle, then

$$L^2 - 12\sqrt{3}A \geq (L - 6\sqrt{3}r)^2 \tag{2'}$$

$$L^2 - 12\sqrt{3}A \geq \left(\frac{A}{r} - 3\sqrt{3}r\right)^2 \tag{3'}$$

$$L^2 - 12\sqrt{3}A \geq \left(L - \frac{2A}{r}\right)^2 \tag{4'}$$

and in each case equality holds if and only if the triangle is equilateral.

Proof. By Osserman's Lemma (changing ρ to r and π to $3\sqrt{3}$), we see that inequalities (2′), (3′), and (4′) are equivalent. It is well known [8], that for a triangle, $A = rs$ where s denotes the semiperimeter ($L/2$). Thus, inequality (4′) is equivalent to (1′) because $L - 2A/r = L - 2s = 0$. Hence, all three inequalities are valid.

Inequality (2′) can be found in [9].

The standard Bonnesen inequalities have the property that each side of the inequality is 0 when the convex figure is a disc. In our triangle analogs, we want each side of the inequality to be 0 when the triangle is equilateral. Thus, the triangle analog of "$L^2 - 4\pi A$" is "$L^2 - 12\sqrt{3}A$".

Inequality (5) has the expression "$2\pi R - L$" on the right, which is 0 for a circle. The analog for triangles is "$3\sqrt{3}R - L$" since it is known that for all triangles, $3\sqrt{3}R \geq L$ with equality when and only when the triangle is equilateral [6]. However, there is no Bonnesen–style analog of the form

$$L^2 - 12\sqrt{3}A \geq k(3\sqrt{3}R - L)^2$$

with $k > 0$ for it is straightforward to show that the ratio $(L^2 - 12\sqrt{3}A)/(3\sqrt{3}R - L)^2$ approaches 0 as the triangle approaches a degenerate triangle.

A referee has pointed out that the reason inequalities of the form (5)–(10) do not exist for triangles is because Osserman's inequality

$$xL \geq A + \pi x^2$$

(which is true for convex sets when $r \leq x \leq R$) is not valid for $x = R$ when π is replaced by $3\sqrt{3}$.

Instead, we have the following analog to inequality (5).

Theorem 2. If A, L, r, and R denote the area, perimeter, inradius, and circumradius of a triangle, then

$$L^2 - 12\sqrt{3}A \geq \frac{64}{9}\sqrt{3}\,r(3\sqrt{3}R - L) \tag{5'}$$

with equality when and only when the triangle is either an equilateral triangle or a degenerate isosceles triangle (with sides a, a, and 2a).

The equality case is straightforward. If the sides of a triangle are a, a, and x, then

$$\begin{aligned}
\lim_{x\uparrow 2a} \frac{L^2 - 12\sqrt{3}A}{r(3\sqrt{3}R - L)} &= \lim_{x\uparrow 2a} \frac{L^2 - 12\sqrt{3}A}{(3\sqrt{3}Rr - rL)} \\
&= \lim_{x\uparrow 2a} \frac{(2a+x)^2 - 12\sqrt{3}A}{(3\sqrt{3}\frac{a^2x}{2L} - 2A)} \\
&= \frac{(4a)^2 - 0}{3\sqrt{3}\frac{a^2(2a)}{8a} - 0} \\
&= \frac{64}{9}\sqrt{3}.
\end{aligned}$$

We have used above the facts that in a triangle, $r = 2A/L$ and $R = abc/4A$ [8].

To prove the inequality in general will require some machinery. Before proceeding to the proof, we review the proof technique devised by Blundon [1]. Other expositions of this technique can be found in [7] and [11].

Given an ordered triple (R, r, s) of positive real numbers, a triangle with circumradius R, inradius r, and semiperimeter s exists if and only if the triple satisfies Blundon's Fundamental Inequality:

$$s^2(18Rr - 9r^2 - s^2)^2 \leq (s^2 - 3r^2 - 12Rr)^3. \tag{11}$$

This is a homogeneous polynomial in R, r, and s, so only the ratios of R, r, and s are of interest. Following a variation of Bottema [7], we let

$$\begin{aligned}
x &= \frac{r}{R} \\
y &= \frac{s}{R}
\end{aligned} \tag{12}$$

and consider x and y as Cartesian coordinates in the Euclidean plane. Inequality (11) transforms into

$$(x^2+y^2)^2 + 12x^3 - 20xy^2 + 48x^2 - 4y^2 + 64x \leq 0. \tag{13}$$

Each point in the xy–plane corresponds to an equivalence class of triples (R, r, s). Those triples that determine a triangle lie inside the region, K, bounded by the y–axis and the hypocycloid whose parametric representation is given by

$$\begin{aligned} x &= \frac{4t^2(1-t^2)}{(1+t^2)^2} \\ & \qquad\qquad\qquad 0 < t < 1. \\ y &= \frac{8t}{(1+t^2)^2} \end{aligned} \tag{14}$$

The region K has cusps at $(0, 0)$, $(0, 2)$, and $(1/2, 3\sqrt{3}/2)$. The points on the bounding hypocycloid correspond to isosceles triangles. The points of K on the y–axis correspond to degenerate triangles.

To verify a proposed homogeneous inequality between R, r, and s, one need only show that the graph of the proposed inequality in this xy–plane contains the region K.

Proof of Theorem 2. We want to find the largest value of k such that the inequality

$$L^2 - 12\sqrt{3}A \geq kr(3\sqrt{3}R - L) \tag{15}$$

holds for all triangles. Let $L = 2s$ and $A = rs$. Apply the transformation (12) to get

$$f(x, y) = 4y^2 + 2(k - 6\sqrt{3})xy - 3kx\sqrt{3} \geq 0. \tag{16}$$

The graph of $f(x, y) = 0$ is an ellipse and a point satisfies inequality (16) if it lies on or outside this ellipse. We therefore need only show that the region K lies on or outside this ellipse. It will suffice to show that the boundary of K lies on or outside the ellipse. Applying the transformation (14), we see that this sufficiency condition is equivalent to

$$64t^2 + 16(k - 6\sqrt{3})t^3(1 - t^2) - 3k\sqrt{3}(1 + t^2)^2t^2(1 - t^2) \geq 0$$

for $0 < t < 1$.

This is equivalent to

$$k \leq \frac{64 - 96\sqrt{3}t(1-t^2)}{(1-t^2)\left[3\sqrt{3}(1+t^2)^2 - 16t\right]}.$$

Let $z = t\sqrt{3}$ to get

$$\frac{k}{96\sqrt{3}} \leq \frac{2 - z(3-z^2)}{(3-z^2)\left[(3+z^2)^2 - 16z\right]}$$

for $0 < z < \sqrt{3}$. Invert and cancel the common factor $(z-1)^2$ and we get:

$$\frac{96\sqrt{3}}{k} \geq \frac{(3-z^2)(z^2+2z+9)}{z+2} = -z^3 - 6z + 18 - \frac{9}{z+2} \equiv h(z). \tag{17}$$

We are looking for the largest value of k for which the inequality (17) holds for all z in the interval $(0, \sqrt{3})$. In other words, we need to determine the maximum value of $h(z)$ for $z \in (0, \sqrt{3})$. It is straightforward to verify that, in the interval $(0, \sqrt{3})$, $h(z)$ monotonically decreases from $27/2$ to 0, so the maximum value of $h(z)$ is $27/2$. We thus see that the largest value that k can have occurs when $96\sqrt{3}/k = 27/2$, i.e. when $k = 64\sqrt{3}/9$.

Equality holds when and only when $z = 0$ or equivalently, $(x, y) = (0, 0)$. The point $(0, 0)$ of region K corresponds to degenerate isosceles triangles.

A more remarkable theorem comes about as the analog of inequality (6). Again, the term "$(R - r)$" should be replaced by "$(R - 2r)$" since it is well known that for all triangles, $R \geq 2r$ with equality when and only when the triangle is equilateral [6]. It is also straightforward to show that there is no analog of the form

$$L^2 - 12\sqrt{3}A \geq k(R - 2r)^2$$

with $k > 0$ because the ratio $(L^2 - 12\sqrt{3}A)/(R - 2r)^2$ approaches 0 as the triangle approaches an equilateral triangle. We have, however, the following analog:

<u>Theorem 3</u>. If A, L, r, and R denote the area, perimeter, inradius, and circumradius of a triangle, then

$$L^2 - 12\sqrt{3}A \geq \mu \;\, r(R - 2r) \tag{8$'$}$$

where $\mu \approx 35.098131$ is a root of the equation $w^3 - 280w^2 + 10368w - 62208 = 0$. Equality holds when and only when either the triangle is equilateral or the triangle is similar to the isosceles triangle with sides 1, 1, and λ where $\lambda \approx 1.23628634$ is the largest root of the equation $31x^3 - 28x^2 - 16x + 4 = 0$.

Proof. We again apply the technique of Blundon. We want to find the largest value of k such that the inequality

$$L^2 - 12\sqrt{3}A \geq kr(R - 2r)$$

holds for all triangles. Let $L = 2s$ and $A = rs$. Apply the transformation (12) to get

$$f(x, y) \equiv 4y^2 + 2kx^2 - 12xy\sqrt{3} - kx \geq 0. \tag{18}$$

The graph of $f(x, y) = 0$ is an ellipse and it suffices to show that the boundary of K lies on or outside this ellipse, i.e. that the boundary of K satisfies inequality (18). Applying the transformation (14) shows that this sufficiency condition is equivalent to

$$256t^2 + 32kt^4(1 - t^2)^2 - 384t^3(1 - t^2)\sqrt{3} - 4kt^2(1 - t^2)(1 + t^2)^2 \geq 0$$

for $0 < t < 1$. Solving for k and letting $z = t\sqrt{3}$ gives

$$k \leq \frac{96(z + 2)}{(z + 1)^2(3 - z^2)} \tag{19}$$

for $0 < z < \sqrt{3}$. We are therefore looking for the largest value of k for which

$$h(z) \equiv -z^3 + 2z + 2 - \frac{1}{z + 2} \leq \frac{96}{k}$$

in the interval $0 < z < \sqrt{3}$. Thus, for this k, $96/k$ is the maximum value of $h(z)$ in the interval $(0, \sqrt{3})$. We note that

$$h'(z) = -3z^2 + 2 + \frac{1}{(z + 2)^2}$$

and that $h'(z) = 0$ if and only if

$$(z + 1)(3z^3 + 9z^2 + z - 9) = 0.$$

This equation is true for only one positive value of z, so $h(z)$ has one relative maximum in the interval $(0, \sqrt{3})$. The value of $h(z)$ at this extremal point, z_0, is larger than the value of $h(0) = 3/2$ or $h(\sqrt{3}) = 0$, so this is the absolute maximum on that interval.

Equality occurs when and only when $z = z_0$, where $z_0 \approx 0.841399865$ is a solution of the equation

$$3z^3 + 9z^2 + z - 9 = 0. \tag{20}$$

Thus k_0, the corresponding value of k, is obtained from the equality condition in inequality (19):

$$k_0 = \frac{96(z_0 + 2)}{(z_0 + 1)^2(3 - z_0^2)} \approx 35.0981313.$$

It is straightforward to check that

$$\frac{96(z + 2)}{(z + 1)^2(3 - z^2)} \equiv -4(33z^2 + 45z - 70) \pmod{3z^3 + 9z^2 + z - 9}$$

so that

$$k_0 = -4(33z_0^2 + 45z_0 - 70).$$

It is also straightforward to check that

$$k_0^3 - 280k_0^2 + 10368k_0 - 62208 \equiv 0 \pmod{3z^3 + 9z^2 + z - 9}$$

showing that k_0 is a root of the equation $w^3 - 280w^2 + 10368w - 62208 = 0$ as claimed.

Note that equality occurs on the boundary of K, i.e. when the triangle is isosceles. As an aside, had we known this in advance, we could have proceeded as follows.

Assume that the triangle that achieves the minimum value of

$$\frac{L^2 - 12\sqrt{3}A}{r(R - 2r)}$$

has sides 1, 1, and x. Then

$$k = \frac{L^2 - 12\sqrt{3}A}{r(R - 2r)} = 2(x + 2)^2 \frac{(x + 2)\sqrt{4 - x^2} + 3x(x - 2)\sqrt{3}}{(x - 1)^2 x\sqrt{4 - x^2}} \equiv f(x)$$

where x can vary from 0 to 2. It is straightforward to calculate that

$$\lim_{x \downarrow 0} f(x) = \infty$$

and

$$\lim_{x \uparrow 2} f(x) = 64,$$

so the minimum value of f does not occur at an endpoint of the interval $(0, 2)$. Also,

$$\lim_{x \to 1} f(x) = 36 > 35.098,$$

so the minimum does not occur at $x = 1$. The minimum must therefore occur at a point where $f'(x) = 0$. Taking the derivative, we find that

$$f'(x) = -4(x+2)\frac{(x+2)(4x-1)\sqrt{4-x^2} + 3\sqrt{3}x^2(2x-5)}{(x-1)^3 x^2 \sqrt{4-x^2}}.$$

The derivative vanishes if

$$(x+2)(4x-1)\sqrt{4-x^2} = 3\sqrt{3}x^2(5-2x)$$

since $x = -2$ is of no concern to us and $x = 1$ and $x = 2$ have already been ruled out. Squaring both sides gives

$$(x+2)^2(4x-1)^2(4-x^2) = 27x^4(5-2x)^2.$$

Bringing all terms to the same side and factoring gives

$$4(x-1)^3(31x^3 - 28x^2 - 16x + 4) = 0.$$

The value $x = 1$ has already been ruled out, so we see that the minimum must occur when x is a zero of $31x^3 - 28x^2 - 16x + 4$. This polynomial has three zeroes, $x \approx -0.53$, $x \approx 0.197$, and $x \approx 1.2362863384$. The first zero is ruled out because it is negative and the second zero is ruled out because it produces a larger value for $f(x)$ than the third zero.

Part of Theorem 1 states that $L^2 - 12\sqrt{3}A \geq (A/r - 3\sqrt{3}r)^2$ holds for all triangles with equality for equilateral triangles. This fact alone does not rule out a stronger inequality of the form $L^2 - 12\sqrt{3}A \geq k(A/r - 3\sqrt{3}r)^2$ for some $k > 1$

since both sides of this inequality are 0 when the triangle is equilateral. In fact, this inequality is true for $k = 4$ since in that case it is equivalent (using $A = rs$) to inequality ($2'$). Applying Blundon's method shows, furthermore, that $k = 4$ yields the best possible inequality of this form. Various other possible analogs of the Bonnesen inequalities (2)–(10) were investigated by this method. Since the proof techniques are no different than those shown in the proofs of Theorems 2 and 3, the tedious details will be omitted and the results are stated as Theorem 4.

Theorem 4. In the Bonnesen–style inequalities

$$L^2 - 12\sqrt{3}A \geq k_1(L - 6\sqrt{3}r)^2 \tag{$1''$}$$

$$L^2 - 12\sqrt{3}A \geq k_2\left(\frac{A}{r} - 3\sqrt{3}r\right)^2 \tag{$2''$}$$

$$L^2 - 12\sqrt{3}A \geq k_3r(L - 6\sqrt{3}r) \tag{$3''$}$$

$$L^2 - 12\sqrt{3}A \geq k_4r(3\sqrt{3}R - L) \tag{$4''$}$$

$$L^2 - 12\sqrt{3}A \geq k_5r(R - 2r) \tag{$5''$}$$

$$L^2 - 12\sqrt{3}A \geq k_6r\left(\frac{3}{4}\sqrt{3}R - \frac{A}{R}\right) \tag{$6''$}$$

$$L^2 - 12\sqrt{3}A \geq k_7\left(L - \frac{4A}{R}\right)^2 \tag{$7''$}$$

$$L^2 - 12\sqrt{3}A \geq k_8A^2\left(\frac{1}{2r} - \frac{1}{R}\right)^2 \tag{$8''$}$$

$$L^2 - 12\sqrt{3}A \geq k_9L^2\frac{(R - 2r)^2}{(R + 2r)^2} \tag{$9''$}$$

$$L^2 - 12\sqrt{3}A \geq k_{10}r\left(\frac{A}{r} - 3\sqrt{3}r\right) \tag{$10''$}$$

$$L^2 - 12\sqrt{3}A \geq k_{11}r\left(L - \frac{4A}{R}\right) \tag{$11''$}$$

1. The best possible triangle inequality of the form $(1'')$ occurs when $k_1 = 1$. In that case, equality occurs when and only when the triangle is either an equilateral triangle or a degenerate isosceles triangle. This inequality is equivalent to $L \geq 6\sqrt{3}r$.
2. Inequality $(2'')$ is equivalent to inequality $(1'')$ with $k_2 = 4k_1$.
3. The best possible triangle inequality of the form $(3'')$ occurs when $k_3 = 6\sqrt{3}$. In that case, equality occurs when and only when the triangle is equilateral. Inequality $(3'')$ is equivalent to the condition $\{L = 6\sqrt{3}r \text{ or } L \geq k_3 r\}$.
4. The best possible triangle inequality of the form $(4'')$ occurs when $k_4 = 64\sqrt{3}/9$. In that case, equality occurs when and only when the triangle is either an equilateral triangle or a degenerate isosceles triangle. This is the same as Theorem 2 but is restated here to show the correspondence between inequalities (2)–(10) and $(1'')$-$(9'')$.
5. The best possible triangle inequality of the form $(5'')$ occurs when $k_5 \approx 35.0981313$ is the second largest root of the equation $x^3 - 280x^2 + 10368x - 62208 = 0$. In that case, equality occurs when and only when either the triangle is equilateral or the triangle is similar to the triangle with sides 1, 1, and λ where $\lambda \approx 1.23628634$ is the largest root of the equation $31x^3 - 28x^2 - 16x + 4 = 0$. This is the same as Theorem 3.
6. The best possible triangle inequality of the form $(6'')$ occurs when $k_6 \approx 19.9777234$. In that case, equality occurs when and only when either the triangle is equilateral or the triangle is similar to the triangle with sides 1, 1, and λ where $\lambda \approx 1.23983866$ is the smallest real root of the equation $7x^8 + 45x^7 + 60x^6 - 162x^5 - 447x^4 - 99x^3 + 488x^2 + 324x + 108 = 0$.
7. The best possible triangle inequality of the form $(7'')$ occurs when $k_7 \approx 0.87281834$. In that case, equality occurs when and only when either the triangle is equilateral or the triangle is similar to the triangle with sides 1, 1, and λ where $\lambda \approx 1.956272$ is the largest root of the equation $28x^4 + 10x^3 - 69x^2 - 94x - 37 = 0$.
8. Inequality $(8'')$ is equivalent to inequality $(7'')$ with $k_8 = 16k_7$.
9. The best possible triangle inequality of the form $(9'')$ occurs when $k_9 \approx 0.94204112$. In that case, equality occurs when and only when either the triangle is equilateral or the triangle is similar to the triangle with sides 1, 1, and λ where $\lambda \approx 1.9913932$ is the largest real root of the equation $27x^6 - 54x^5 + 193x^4 - 392x^3 + 354x^2 - 538x - 229 = 0$.
10. Inequality $(10'')$ is equivalent to inequality $(3'')$ with $k_{10} = 2k_3$.
11. The best possible triangle inequality of the form $(11'')$ occurs when $k_{11} \approx 6.829212$. In that case, equality occurs when and only when either the triangle is equilateral or the triangle is similar to the triangle with sides 1, 1, and λ where $\lambda \approx 1.129475$ is the second largest real root of the equation $7x^4 - 18x^3 + 5x^2 + 9x - 2 = 0$.

There are no triangle inequalities with any of the forms

$$L^2 - 12\sqrt{3}A \geq k(3\sqrt{3}R - L)^2$$

$$L^2 - 12\sqrt{3}A \geq k(R - 2r)^2$$

$$L^2 - 12\sqrt{3}A \geq k\left(\frac{3}{4}\sqrt{3}R - \frac{A}{R}\right)^2$$

with $k > 0$.

Note. The forms considered as possible analogs of the Bonnesen inequalities have the property that the left side of the inequality represents the "isoperimetric deficit". Both sides of the inequality should be 0 for the equilateral triangle. In justifying the forms considered above, we point out the following known inequalities (with equality when and only when the triangle is equilateral): $R \geq 2r$, $3\sqrt{3}R \geq L$, $L \geq 6\sqrt{3}r$, and $3\sqrt{3}r^2 \leq A \leq \frac{3}{4}\sqrt{3}R^2$ [6].

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