USING WHALES TO COMPLETE A BOOLEAN ALGEBRA

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In this paper, $\mathbb{B}$ will denote an arbitrary Boolean algebra. We use certain "large" subsets of $\mathbb{B}$, known as whales, to construct an Archimedean Riesz space, also known as a vector lattice, which contains a natural, complete Boolean algebra. The latter Boolean algebra, known as the Boolean algebra of bands, will be used as the target for the completion of $\mathbb{B}$. In our concluding remarks, we make a connection between the completion of $\mathbb{B}$ and Stone's Representation Theorem. It is known that every Boolean algebra has a completion. Moreover, a Boolean algebra is complete if and only if it is isomorphic to the regular open algebra of some topological space [2].

Whales were introduced in [1] and employed to give a short proof of the universal completion of an Archimedean Riesz space. We adapt the techniques in [1] to the setting of Boolean algebras and give a simple, constructive proof that any Boolean algebra can be represented as a dense subalgebra of the Boolean algebra of all bands of an Archimedean Riesz space. For an excellent text on elementary Riesz space theory, we refer the reader to [5].

In what follows, we denote the partial order of $\mathbb{B}$ by $\leq$, the largest element by $\mathbf{1}$, and the smallest element by $\mathbf{0}$. If for every subset D of $\mathbb{B}$, the least upper bound and the greatest lower bound of D exist, then $\mathbb{B}$ is *complete.* A *completion* of $\mathbb{B}$ is a complete Boolean algebra having $\mathbb{B}$ as a dense subalgebra. If $B \in \mathbb{B}$, then B' will denote the complement of B. A subset $\mathcal{A}$ of a Boolean algebra $\mathbb{B}$ is called a *whale* if

1. For every $A \in \mathcal{A}$ and every $B \in \mathbb{B}$ with $B \leq A$, we have $B \in \mathcal{A}$, and
2. $\sup\{A : A \in \mathcal{A}\} = \mathbf{1}$.

It is easy to see that any union of whales is a whale. Also, if $\mathcal{A}$ and $\mathcal{B}$ are whales, then the set $\mathcal{A} \wedge \mathcal{B} := \{A \wedge B : A \in \mathcal{A},\ B \in \mathcal{B}\}$ is a whale.

1. Constructing a Riesz Space. Associated to each whale $\mathcal{A}$, we define

$$\mathbb{R}_{\mathcal{A}} := \Big\{ f : \mathcal{A} \to \mathbb{R} \text{ such that } f(\mathbf{0}) = 0 \text{ and } f(B) = f(A)$$

$$\text{whenever } B \leq A,\ B \neq \mathbf{0} \Big\}.$$

Under the usual pointwise addition and order, each $\mathbb{R}_{\mathcal{A}}$ is a Dedekind complete Riesz space. If $f \in \mathbb{R}_{\mathcal{A}}$, we write dom f for $\mathcal{A}$. Define a binary relation on $\bigcup\{\mathbb{R}_{\mathcal{A}}\ :\ \mathcal{A} \text{ is a whale}\}$ by

$$f \sim g := \{f(A) = g(A) \text{ for all } A \in \text{dom } f \wedge \text{dom } g\}.$$

To see that $\sim$ is indeed an equivalence relation, suppose that $f \sim g$ and $g \sim h$. Let $A \in \text{dom } f \wedge \text{dom } h$. Then for all $B \in \text{dom } g$,

$$f(A) = f(A \wedge B) = g(A \wedge B) = h(A \wedge B) = h(A).$$

That $\sim$ is reflexive and symmetric is obvious. We state the most important fact about $\sim$ in the next lemma. For a proof of Lemma 2, see [1].

Lemma 1. Each equivalence class contains a unique element with maximal domain.

If $f \in \cup\{\mathbb{R}_{\mathcal{A}}\ :\ \mathcal{A} \text{ is a whale}\}$, then we denote the equivalence class of f by $[f]$ and the unique element of $[f]$ with maximal domain by $\overline{f}$. We then denote the set of all elements with maximal domains by $E(\mathbb{B})$. If $\overline{f}, \overline{g} \in E(\mathbb{B})$, then one can find an $\overline{h}$ for which $\overline{h} = \overline{f} + \overline{g}$ as follows. Define an element $h \in \mathbb{R}_{\text{dom } \overline{f} \wedge \text{dom } \overline{g}}$ by $h(A) = \overline{f}(A) + \overline{g}(A)$ $(A \in \text{dom } \overline{f} \wedge \text{dom } \overline{g})$. By Lemma 2, $\overline{h}$ exists and is unique. One routinely proves that $E(\mathbb{B})$ with this addition and the obvious scalar multiplication is a real vector space. We equip $E(\mathbb{B})$ with an order relation given by $f \leq g$ if $f(A) \leq g(A)$ for all $A \in \text{dom } f \wedge \text{dom } g$. This order relation enables us to prove the following theorem.

Theorem 2. $E(\mathbb{B})$ is an Archimedean Riesz space.

Proof. Lemma 1 guarantees that the order defined above is indeed a partial order. For example, if $\overline{f} \leq \overline{g}$ and $\overline{g} \leq \overline{f}$, then $\overline{f} \sim \overline{g}$. It follows then by Lemma

2 that $\overline{f} = \overline{g}$. We prove that the supremum of any two elements in $E(\mathbb{B})$ exists and leave the rest of the details to the reader. Let $\overline{f}, \overline{g} \in E(\mathbb{B})$. Define an element $h \in \mathbb{R}_{\text{dom } \overline{f} \wedge \text{dom } \overline{g}}$ by

$$h(A) = \max\{\overline{f}(A), \overline{g}(A) \quad (A \in \text{dom } \overline{f} \wedge \text{dom } \overline{g})\}.$$

By Lemma 1, we obtain $\overline{h} \in E(\mathbb{B})$. Since $\overline{h} \sim h$ and dom $h \subset$ dom $\overline{h}$, it follows that $\overline{h} \geq \overline{f}$ and $\overline{h} \geq \overline{g}$. Suppose that $\overline{y} \geq \overline{f}$, $\overline{y} \geq \overline{g}$, and $A \in \text{dom } \overline{y} \wedge \text{dom } \overline{h}$ such that $A \neq \mathbf{0}$. Then

$$\overline{y}(A) = \overline{y}(A \wedge B) \geq \overline{h}(A \wedge B) = \overline{h}(A).$$

Therefore, $\overline{h} = \overline{f} \vee \overline{g}$.

2. Characteristic Functions. Every $B \in \mathbb{B}$ gives rise to a whale $\mathcal{A}_B = \{A \leq B\} \cup \{A \leq B'\}$. Define a function $\chi_B \in \mathbb{R}_{\mathcal{A}_B}$ by

$$\chi_B(A) := \{1 \text{ if } A \leq B, \quad 0 \text{ if } A \leq B'\}.$$

It is easy to see that χ_B is an element with maximal domain and thus a member of $E(\mathbb{B})$. We write $B[E(\mathbb{B})]$ for the Boolean algebra of all bands of $E(\mathbb{B})$. If $f \in E(\mathbb{B})$, then f' is the band given by $\{g \in E(\mathbb{B}) : |g| \wedge |f| = 0\}$. The characteristic functions give us a natural way to define a map between the Boolean algebras $\mathbb{B}$ and $B[E(\mathbb{B})]$. We summarize the important properties of that map in Theorem 4. Before proving Theorem 4, we need the following lemma.

Lemma 3. Let B and C be elements of $\mathbb{B}$. Then $(\chi_B)'' \cap (\chi_C)'' = (\chi_B \wedge \chi_C)''$.

Proof. Since $\chi_{B \wedge C} = \chi_B \wedge \chi_C \in (\chi_B)'' \cap (\chi_C)''$, it follows that $(\chi_B \wedge \chi_C)'' \subset (\chi_B)'' \cap (\chi_C)''$. For the other inclusion, let $f \in (\chi_B)'' \cap (\chi_C)''$ and $g \in (\chi_B \wedge \chi_C)'$. This implies that $g \in (\chi_B)' \cap (\chi_C)'$. Thus, $|f| \wedge |g| = 0$ and $f \in (\chi_B \wedge \chi_C)''$. Hence, $(\chi_B \wedge \chi_C)'' = (\chi_B)'' \cap (\chi_C)''$.

Theorem 4. Let $\varphi\colon \mathbb{B} \to B[E(\mathbb{B})]$ be given by $B \mapsto (\chi_B)''$. Then φ is an injective Boolean homomorphism. Moreover, $\varphi(\mathbb{B})$ is a dense subalgebra of $B[E(\mathbb{B})]$.

Proof. Applying the identity $(\chi_{B'})' = (\chi_B)''$, together with Lemma 3, we get that φ is a Boolean homomorphism. If $B \in \mathbb{B}$ and $B \neq \mathbf{0}$, then χ_B is a nonzero element of $(\chi_B)''$. Thus, φ is injective. For the density, let $D \neq \{0\}$ be an element of $B[E(\mathbb{B})]$ and let $0 < f \in D$. There exists $A \in \text{dom } f$ such that $f(A) = k > 0$.

If $C \leq A$, then $k \cdot \chi_A(C) = k = f(C)$. If $C \leq A'$, then $k \cdot \chi_A(C) = 0 \leq f(C)$. It follows that $k \cdot \chi_A \leq f$ and

$$(k \cdot \chi_A)'' = (\chi_A)'' \subset f'' \subset D.$$

The Boolean algebra $B[E(\mathbb{B})]$ is complete [3]. The latter fact, in conjuction with Theorem 4.19 in [2] and Theorem 4, yields the next result.

Theorem 5. The Boolean algebra $B[E(\mathbb{B})]$ is the completion of $\mathbb{B}$.

Since the completion of a Boolean algebra can be represented as the bands of an Archimedean Riesz space, the following corollary is immediate.

Corollary 6. Every Boolean algebra is isomorphic to a dense subalgebra of the Boolean algebra of all bands of an Archimedean Riesz space.

3. A Connection With Stone's Theorem. Perhaps the most important result regarding the representation of a Boolean algebra is the topological version of Stone's Representation Theorem. We state this as Theorem 7 below. For a proof see [2] or [4].

Theorem 7. [Stone's Representation Theorem]. Every Boolean algebra is isomorphic to the algebra of closed-open sets of a totally disconnnected, compact Hausdorff space.

The topological space mentioned in Theorem 7 is called the *Stone space* of the Boolean algebra. If a Boolean algebra is complete, then the corresponding Stone space is extremally disconnected [4]. It is straightforward, using the techniques in [1], to prove that $E(\mathbb{B})$ is a universally complete Riesz space. Applying the Ogasawara-Maeda representation theorem, $E(\mathbb{B})$ is of the form $C^{\infty}(X)$ for some compact, extremely disconnected Hausdorff space X. For a detailed account of this fact, see [3]. The Stone space of $B[E(\mathbb{B})]$ is precisely the topological space X. Thus, the Boolean algebra $B[E(\mathbb{B})]$ corresponds to the Boolean algebra of closed-open sets of X. The original Boolean algebra $\mathbb{B}$ corresponds to a basis of closed-open sets of X.

References

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