RELATIVE ALGEBRAIC STRUCTURES

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Abstract. The concept and some of the algebraic properties of the rejective and non–absorptive sets of a subgroup, subring, and subgroup of a module over a ring are investigated. It is shown that the set theoretic complement of a non–absorptive set in the above mentioned algebraic substructures is a normal subgroup (respectively, (left, right) ideal, submodule) of its underlying algebraic structure. The invariant property of the non–absorptive sets under the operation of inversion in the related underlying algebraic structure is proved. $G \setminus R(H)$, the set theoretic complement of the rejective set of a subgroup H in a group G, is closed under the product in G and whenever $|G|$ the order of the group G is finite, $|R(H)| = (k-s)|H|$ where each of the k and s is the index of H in G and in $G \setminus R(H)$, respectively. For the case of rings and modules, the set theoretic complement of the rejective set of a substructure in the underlying ring is a subring of the underlying ring. For any subring S of a ring R, examples and some of the properties of S–relative (left) ideals and S–relative submodules are given and also it is shown that S is contained in the set theoretic complement of the rejective set of that S–relative (left) ideal (respectively, submodule). Finally, some of the properties of the relative homomorphisms of R–modules, and the rejective (respectively, non–absorptive) sets of the group homomorphisms of R–modules are investigated.

1. Rejective and Non–Absorptive Sets of Some Algebraic Substructures.

Definition 1.1. For a subgroup H of a group G, the set of all elements h in H such that for each h there exists an element g in G with $ghg^{-1} \notin H$ is called the non–absorptive set of H in G and is denoted by $N(H, G)$ or $N(H)$ whenever there is no confusion in the context.

Remark. In the above definition, it is clear that H is a normal subgroup of G if and only if $N(H)$ is the empty set.

Theorem 1.1. Let $N(H)$ be the non–absorptive set of a subgroup H in a group G. Then $H \setminus N(H)$ is a normal subgroup of G.

Proof. Suppose to the contrary that a and b are in $H \setminus N(H)$ and ab is in $N(H)$. Thus, for some $g \in G$, $gabg^{-1} = gag^{-1}gbg^{-1} \notin H$ which is a contradiction since both gag^{-1} and gbg^{-1} are in H. Next, $b \in H \setminus N(H)$ implies $gbg^{-1} \in H$ for all $g \in G$ and this makes $(gbg^{-1})^{-1} = gb^{-1}g^{-1}$ to be in H for all $g \in G$ which implies $b^{-1} \in H \setminus N(H)$. Finally, it remains to show that $H \setminus N(H)$ is normal in G. Suppose for some $a \in H \setminus N(H)$ there exists $g \in G$ such that $gag^{-1} \notin H \setminus N(H)$. Hence, by definition, $gag^{-1} \in N(H)$ and this forces $k(gag^{-1})k^{-1} = (kg)a(kg)^{-1} \notin H$ for some k in G and this is a contradiction to the choice of a in $H \setminus N(H)$.

Theorem 1.2. Let $N(H)$ be the non–absorptive set of a subgroup H in a group G. Then for each $x \in G \setminus N(H)$, x^{-1} is also in $G \setminus N(H)$.

Proof. Suppose to the contrary that $x \in G \setminus N(H)$ and x^{-1} is in $N(H)$. Consequently, x is in $H \setminus N(H)$ which implies $(gxg^{-1})^{-1} = gx^{-1}g^{-1} \in H$ for all $g \in G$ and this is a contradiction to the choice of x^{-1} in $N(H)$.

Remark. From the above theorem, it is clear that $x^{-1} \in N(H)$ whenever $x \in N(H)$. In other words, $N(H)$ is invariant under the group operation of inversion. Furthermore, $N(H)$ can never be closed under the product in the group since e the group identity element is not in $N(H)$.

Corollary 1.1. Let $N(H)$ be the non–absorptive set of a subgroup H in a group G. Then $G \setminus N(H)$ is a subgroup of G if and only if $G \setminus N(H)$ is closed under the product in G.

Remark. For a family $\{G_i \mid i \in I\}$ of groups with H_i a subgroup of G_i for each $i \in I$, it is not difficult to show that $\prod N(H_i, G_i) \subseteq N(\prod H_i, \prod G_i)$ and for $|I| = 2$, $(N(H_1) \times H_2) \cup (H_1 \times N(H_2)) = N(H_1 \times H_2)$. In general, $\cup_{i \in I} \prod^j H_i = N(\prod H_i)$ where $\prod^j H_i$ is the Cartesian product of all subgroups H_i $(i \neq j)$ and $N(H_j)$ for $i = j$.

Definition 1.2. For a subgroup H of a group G, the set of all $g \in G$ such that for each g there exists an element h in H with $ghg^{-1} \notin H$ is called the rejective set of H in G and is denoted by $R(H, G)$ or $R(H)$ whenever there is no confusion in the context.

Remark. It is clear that $R(H, G) = \emptyset$ if and only if H is a normal subgroup of G. From the definition, it is obvious that $R(H, G) \subseteq G \setminus H$.

Theorem 1.3. For any subgroup H of a group G, $G \setminus R(H)$ the set theoretic complement of the rejective set of H in G is closed under the group operation of product.

Proof. Suppose to the contrary that $a, b \in G \setminus R(H)$ and $ab \in R(H)$. Thus, for some h in H, $abhb^{-1}a^{-1} = a(bhb^{-1})a^{-1} \notin H$ and this is a contradiction.

Corollary 1.2. Let $R(H)$ be the rejective set of a subgroup H in a group G. Then $G \setminus R(H)$ is a subgroup of G if and only if $R(H)$ is invariant under the operation of inversion in G.

Corollary 1.3. Let $R(H)$ be the rejective set of a subgroup H in a finite group G. Then $G \setminus R(H)$ is a subgroup of G and $|R(H)| = (k-s)|H|$ where k is the index of H in G and s is the index of H in $G \setminus R(H)$.

Proof. Since every non–empty finite subset of a group G is a subgroup of G if and only if it is closed under the product in G, consequently, by applying the above theorem together with Lagrange's Theorem, $G \setminus R(H)$ is a subgroup of G and each of $|G \setminus R(H)|$ and $|G|$ is divisible by $|H|$. Thus, $s|H| = |G \setminus R(H)| = |G| - |R(H)| = k|H| - |R(H)|$ which implies $|R(H)| = (k-s)|H|$ where s is the index of H in $G \setminus R(H)$ and k is the index of H in G.

Example. As an application of the above corollary, it is easy to conclude that any subgroup H of a group G with a finite order $2n$ is normal in G whenever the order of H is n and there exists an element g in $G \setminus H$ such that $ghg^{-1} \in H$ for all $h \in H$.

Definition 1.3. For a subring A of a ring R, the set of all elements $a \in A$ such that for each a there exists an element r in R with $ra \notin A$ (respectively, $ar \notin A$) is called the left (respectively, right) non–absorptive set of A in R and is denoted by $N_l(A, R)$ or $N_l(A)$ (respectively, $N_r(A, R)$ or $N_r(A)$) whenever there is no confusion in the context.

Remark. From the above definition, it is clear that A is a left (respectively, right) ideal in R if and only if $N_l(A)$ (respectively, $N_r(A)$) is the empty set.

Theorem 1.4. Let A be a subring of a ring R. Then $A \setminus N_l(A)$ (respectively, $A \setminus N_r(A)$) is a left (respectively, right) ideal of R.

Proof. Let each of a and b be an element in $A \setminus N_l(A)$ and suppose to the contrary that $(a-b) \notin A \setminus N_l(A)$. Then for some r in R, $r(a-b) = ra - rb \notin A$

which is a contradiction since both ra and rb are in A. Now, for any $a \in A \setminus N_l(A)$ and $r \in R$ if $ra \notin A \setminus N_l(A)$, then ra must be in $N_l(A)$. Hence, for some s in R, $s(ra) = (sr)a \notin A$ which is a contradiction to the choice of a in $A \setminus N_l(A)$. A proof for the case of $N_r(A)$ can be followed analogously.

Theorem 1.5. Let $N_l(A)$ be the left non–absorptive set of a subring A in a ring R. Then for each element $a \in R \setminus N_l(A)$, $-a$ is also in $R \setminus N_l(A)$. In other words, $a \in N_l(A)$ implies $-a \in N_l(A)$.

Proof. Suppose $a \in R \setminus N_l(A)$ and $-a \in N_l(A)$. Then a must be in $A \setminus N_l(A)$ since A is an additive subgroup of R. Thus, $r(-a) = -r(a) \in A$ for all r in R and this is a contradiction to the choice of $-a \in N_l(A)$.

Remark. Let $\{R_i \mid i \in I\}$ be a family of rings with A_i a subring of R_i for each $i \in I$. Then $\prod N_l(A_i, R_i) \subseteq N_l(\prod A_i, \prod R_i)$, and for $|I| = 2$, we have $(N_l(A_1) \times A_2) \cup (A_1 \times N_l(A_2)) = N_l(A_1 \times A_2)$. In general, $\cup_{i \in I} \prod^j A_i = N_l(\prod_{i \in I} A_i)$ where $\prod^j A_i$ is the Cartesian product of all subrings A_i $(i \neq j)$ and $N_l(A_j)$ for $i = j$.

Remark. In any commutative ring, it is obvious that the left and right non–absorptive sets of a subring coincide with each other.

Definition 1.4. For a subring A of a ring R, the set of all $r \in R$ such that for each r there exists an element a in A with $ra \notin A$ (respectively, $ar \notin A$) is called the left (respectively, right) rejective set of A in R and is denoted by $R_l(A, R)$ or $R_l(A)$ (respectively, $R_r(A, R)$ or $R_r(A)$) whenever there is no confusion in the context.

Remark. It is clear that $R_l(A)$ (respectively, $R_r(A)$) is a subset of $R \setminus A$ and A is a left (respectively, right) ideal of R if and only if $R_l(A)$ (respectively, $R_r(A)$) is the empty set. Note that in a commutative ring R, both the left and right rejective sets of any subring A of R coincide with each other.

Theorem 1.6. Let $R_l(A)$ (respectively, $R_r(A)$) be the left (respectively, right) rejective set of a subring A in a ring R. Then $R \setminus R_l(A)$ (respectively, $R \setminus R_r(A)$) is a subring of R.

Proof. Suppose $r, s \in R \setminus R_l(A)$ and $r - s \notin R \setminus R_l(A)$. Then for some a in A, $(r - s)a = ra - sa \notin A$ and this is a contradiction since both ra and sa are in A. Now, suppose for some $r, s \in R \setminus R_l(A)$, rs is not in $R \setminus R_l(A)$. Thus, for some a in A, $(rs)a = r(sa) \notin A$ and this is a contradiction to the choice of r and s.

Defintion 1.5. Let A be a subgroup of an R–module M over a ring R. The set of all elements $a \in A$ such that for each a there exists an element $r \in R$ with $ra \notin A$ is called the non–absorptive set of A in M and is denoted by $N(A, M)$ or $N(A)$ whenever there is no confusion in the context.

Remark. From the above definition, it is clear that A is a submodule of M if and only if $N(A)$ is the empty set.

Theorem 1.7. Let $N(A)$ be the non–absorptive set of a subgroup A in an R–module M over a ring R. Then $A \setminus N(A)$ is a submodule of M.

Proof. Similar to the proof of Theorem 1.4.

Theorem 1.8. Let $N(A)$ be the non–absorptive set of a subgroup A in an R–module M over a ring R. Then $a \in M \setminus N(A)$ implies $-a \in M \setminus N(A)$. In other words, $a \in N(A)$ implies $-a \in N(A)$.

Proof. Similar to the proof of Theorem 1.5.

Remark. Let $\{M_i \mid i \in I\}$ be a family of R–module over a ring R and A_i a subgroup of M_i for each $i \in I$. Then $\prod N(A_i) \subseteq N(\prod A_i)$ and for $|I| = 2$, $(N(A_1) \times A_2) \cup (A_1 \times N(A_2)) = N(A_1 \times A_2)$. In general, we have $\cup_{i \in I} \prod^j A_i = N(\prod_{i \in I} A_i)$ where $\prod^j A_i$ is the Cartesian product of all subgroups A_i $(i \neq j)$ and $N(A_j)$ for $i = j$.

Definition 1.6. For a group A of an R–module M over a ring R, the set of all elements $r \in R$ such that for each r there exists an element a in A with $ra \notin A$ is called the rejective set of A in M and is denoted by $R(A, M)$ or $R(A)$ whenever there is no confusion in the context.

Remark. For the above definition, it is clear that A is a submodule of M if and only if $R(A)$ is the empty set.

Theorem 1.9. If $R(A)$ is the rejective set of a subgroup A in an R–module M over a ring R, then $R \setminus R(A)$ is a subring of R.

Proof. Similar to the proof of Theorem 1.6.

Theorem 1.10. For any two subgroups (respectively, subrings, subgroups) A and B of a group (respectively, a ring, an R–module), $A - N(A) \subseteq B - N(B)$ (respectively, $A - N_l(A) \subseteq B - N_l(B)$, $A - N(A) \subseteq B - N(B)$) whenever $A \subseteq B$.

Proof. We just give a proof for the subgroups A and B of a group G and leave the other cases to the reader. Suppose to the contrary that there exists an element $a \in A - N(A)$ with $a \notin B - N(B)$. Thus, $a \in N(B)$ and this implies $gag^{-1} \notin B$ for some $g \in G$. Consequently, gag^{-1} is not in A which implies $a \in N(A)$ and this is a contradiction to the choice of a in $A - N(A)$.

2. Relative Ideals.

Definition 2.1. Let S be a subring of a ring R. A subring A of R is an S–relative left (respectively, right) ideal of R provided $s \in S$ and $a \in A$ imply $sa \in A$ (respectively, $as \in A$). A is an S–relative ideal of R if it is both an S–relative left and an S–relative right ideal of R. A subring A of R is said to be a strictly S–relative left (respectively, right) ideal of R whenever A is an S–relative left (respectively, right) ideal of R and it is not an R–relative left (respectively, right) ideal of R

Remark. Whenever a statement is made about the S–relative left ideals, it is to be understood that the analogous statement holds for the S–relative right ideals. It is clear that any left ideal A of a ring R is an S–relative left ideal of R for any subring S of R. Also, A contains S whenever 1_R the identity element of R is in A.

Example. Let M be the ring of all 2×2 matrices over a ring R. Then A the subring of all matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

is neither a left nor a right ideal of M. Let S be the set of all 2×2 matrices with zero $(2, 1)$ entries and T the set of all 2×2 matrices with zero $(1, 2)$ entries. Now, it is not difficult to show that A is an S–relative left and a T–relative right ideal of M. Furthermore, A is neither an S–relative right ideal nor a T–relative left ideal of M.

Theorem 2.1. If S and A are two subrings of a ring R with A an S–relative left ideal of R, then $R \setminus R_l(A)$ contains S and A is also an $R \setminus R_l(A)$–relative left ideal of R where $R_l(A)$ is the left rejective set of A in R.

Proof. See Theorem 1.6.

Corollary 2.1. For a subring S of a ring R if $\{A_i \mid i \in I\}$ is a family of S–relative left ideals of R, then A_l is a $\cap_{i\in I} R \setminus R_l(A_i)$–relative left ideal of R for each i in I.

Theorem 2.2. The following results can be proved directly from the definition.

a) If A is an S–relative left ideal of a ring R, then $S \cap R_l(A) = \emptyset$.
b) If $f: R \to T$ is a homomorphism of rings and A is an S–relative left ideal of R, then $f(A)$ is an $f(S)$–relative left ideal of T.
c) If A is both an S–relative left and a T–relative right ideal of a ring R, then A is an $S \cap T$–relative ideal of R.
d) Let $\{S_i \mid i \in I\}$ be a family of subrings of a ring R. Then A is a $\cap_{i\in I} S_i$–relative left ideal of R if A is an S_i–relative left ideal of R for each $i \in I$.
e) If $S_1 \subseteq S_2$ are two subrings of a ring R and A is an S_2–relative left ideal of R, then A is an S_1–relative left ideal of R.
f) For any ascending chain $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, A is a $\cup_{i\in I} S_i$–relative left ideal of R if and only if A is an S_i–relative left ideal of R for each $i \in I$.
g) For a family $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, $\cap_{i\in I} A_i$ is a $\cap_{i\in I} S_i$–relative left ideal of R whenever A_i is an S_i–relative left ideal of R for each $i \in I$.
h) Let $\{R_i \mid i \in I\}$ be a family of rings and S_i a subring of R_i for each $i \in I$. If A_i is an S_i–relative left ideal of R_i for each $i \in I$, then $\prod_{i\in I} A_i$ is a $\prod_{i\in I} S_i$–relative left ideal of $\prod_{i\in I} R_i$ the direct product of the rings.

Example. In the ring R of $n \times n$ matrices over a division ring D, let I_k be the set of all matrices that have nonzero entries only in column k and J'_k the set of all matrices with zero kth rows. Then I_k is a left ideal and a J'_k–relative right ideal but not a right ideal of R. If J_k consists of those matrices with nonzero entries only in row k and I'_k the set of all matrices with zero kth columns, then J_k is a right ideal and an I'_k–relative left ideal but not a left ideal in R.

Theorem 2.3. For any subring A of a ring R, the left rejective set of A in R is $R \setminus A$ whenever A contains 1_R the identity element of R.

Proof. $A = R \setminus R_l(A)$ since A is always contained in $R \setminus R_l(A)$ and $1_R \in A$ implies $R \setminus R_l(A) \subseteq A$.

Example. As an application of the above theorem let $R[X]$ be the ring of all polynomials over an integral domain R and A the ring of all polynomials with zero

X–coefficients, then $R(A, R[X])$ the rejective set of A in $R[X]$ is $R[X] \setminus A$ which is exactly the set of all polynomials with nonzero X–coefficients.

Definition 2.2. Let X be a subset of a ring R and S a subring of R. If $\{A_i \mid i \in I\}$ is the family of all S–relative left ideals of R containing X, then $\cap_{i \in I} A_i$ is called the S–relative left ideal generated by X in R and is denoted by $(X)_S$. The elements of X are called S–relative generators of $(X)_S$. If $X = \{x_1, x_2, \dots, x_n\}$, then the S–relative left ideal $(X)_S$ is denoted by $(x_1, x_2, \dots, x_n)_S$ and is said to be an S–relative finitely generated left ideal. An S–relative left ideal $(x)_S$ generated by a single element x is called an S–relative principal left ideal of R.

Theorem 2.4. Let S be a subring of a ring R, a an element in R, and K the set of all elements of the form $ra + as + na + \sum_{i=1}^{m} r_i a s_i$ where $r, s, r_i, s_i \in S$, n an integer, and m runs over the set of non–negative integers. Then we have the following results:

1) $K \subseteq (a)_S$ the S–relative principal ideal generated by a in R. Moreover, $a \in S$ implies $(a)_S = K = (a)^S$ the principal ideal generated by a in S.
2) $x \in (a)_S \setminus K$ implies $-x \in (a)_S \setminus K$.
3) If R is a commutative ring and $a \in S$, then $(a)_S$ consists of all elements of the form $sa + na$ where $s \in S$ and $n \in Z$ the ring of rational integers.

Proof. The proof is an immediate consequence of the definition and we leave it to the reader as an exercise.

Theorem 2.5. For a subring S of a ring R if A is an S–relative ideal of R, then

1) $S + A = \{s + a \mid s \in S,\ a \in A\}$ is also an S–relative ideal in R.
2) $S \cup A$ is a multiplicative system in R.
3) $S \cap A$ is an S–relative ideal of R, and also it is an S–relative left ideal of R whenever A is an S–relative left ideal of R.

Proof. The proof is a direct consequence of the definition and we leave it to the reader.

Theorem 2.6. In a commutative ring R, let each of $S_1, S_2, \dots, S_n$ be a subring of R and A_i an S_i–relative ideal of R for each $i = 1, 2, \dots, n$, respectively. Then $A_1 A_2 A_3 \cdots A_n$ is an $S_{i_1} S_{i_2} \cdots S_{i_k}$–relative ideal of R where $\{i_1, i_2, \dots, i_k\}$ is a subset of the set $\{1, 2, \dots, n\}$.

Proof. The proof follows directly from the definition and we leave it to the reader.

Definition 2.3. Let S be a subring of a ring R. An S–relative left ideal P of R is said to be an S–relative prime left ideal of R if $P \neq R$ and for any S–relative left ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Theorem 2.7. For a subring S of a ring R if P is an S–relative left ideal of R such that $P \neq R$ and for all elements $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$, then P is an S–relative prime left ideal of R. Conversely if R is commutative and P is an S–relative prime ideal of R, then for any $a, b \in S$, $ab \in P$ implies either $a \in P$ or $b \in P$.

Proof. Let A and B be two S–relative left ideals of R such that $AB \subseteq P$. Suppose $A \nsubseteq P$, then there exists an element $a \in A$ with $a \ni P$. Now, for each $b \in B$, $ab \in AB \subseteq P$ implies $a \in P$ or $b \in P$ which implies $b \in P$ and consequently $B \subseteq P$. Conversely, $ab \in P$ implies $(ab)_S \subseteq P$. Now since R is commutative and $a, b \in S$, then $(a)_S(b)_S \subseteq (ab)_S \subseteq P$ which implies the desired conclusion.

Remark. From the above result it is clear that if P is an S–relative prime ideal of a commutative ring R, then $S \setminus P$ is a multiplicative system in R.

Remark. Any S–relative (left) ideal A of a ring R is a (left) S–module.

3. Relative Submodules.

Definition 3.1. For a ring R, let M be an R–module and S a subring of R. A non–empty subset A of M is an S–relative submodule of M provided that A is an additive subgroup of M and $sa \in A$ for all $s \in S$ and $a \in A$.

Example. A subring S of a ring R is an S–relative submodule of R. In general, any S–relative (left) ideal of a ring R is an S–relative submodule of R whenever R is assumed to be an R–module over itself.

Remark. An S–relative submodule B of an R–module A over a ring R need not be a subring of R whenever $A = R$.

Example. Let A be an S–relative left ideal of a ring R and M an R–module. If X is a non–empty subset of M, then $AX = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in A,\ x_i \in X,$ and n a positive integer$\}$ forms an S–relative submodule of M. Similarly, for any $x \in M$, $Ax = \{ax \mid a \in A\}$ is an S–relative submodule of M.

Theorem 3.1. For a subring S of a ring R if A is an S–relative submodule of an R–module M, then A is an $R \setminus R(A)$–relative submodule of M and S is contained in $R \setminus R(A)$ where $R(A)$ is the rejective set of A in M.

Proof. See Theorem 1.9.

Corollary 3.1. For a subring S of a ring R if $\{A_i \mid i \in I\}$ is a family of S–relative submodules of an R–module M, then A_i is a $\cap_{i\in I}(R \setminus R(A_i))$–relative submodule of M for each i in I.

Theorem 3.2. The following results can be proved directly from the definition.

a) If A is an S–relative submodule of a module M over a ring R, then $S \cap R(A) = \emptyset$ where $R(A)$ is the rejective set of A in M.

b) If R is a ring and $f : M \to N$ an R–module homomorphism, then the homomorphic image (respectively, inverse image) of any S–relative submodule of M (respectively, N) is again an S–relative submodule of N (respectively, M).

c) For a ring R if $\{S_i \mid i \in I\}$ is a family of subrings of R, A an R–module, and B_i an S_i–relative submodule of A for each $i \in I$, then $\cap_{i\in I} B_i$ is a $\cap_{i\in I} S_i$–relative submodule of A.

d) If $S_1 \subseteq S_2$ are two subrings of a ring R and A is an S_2–relative submodule of an R–module M, then A is an S_1–relative submodule of M.

e) For any ascending chain $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, A is a $\cup_{i\in I} S_i$–relative submodule of an R–module M if and only if A is an S_i–relative submodule of M for each $i \in I$.

f) For a family $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, $\cap_{i\in I} A_i$ is a $\cap_{i\in I} S_i$–relative submodule of an R–module whenever A_i is an S_i–relative submodule of M for each $i \in I$.

g) For a family of rings $\{R_i \mid i \in I\}$, assume S_i is a subring of R_i, M_i an R_i–module, and A_i an S_i–relative submodule of M_i for each $i \in I$, then $\prod_{i\in I} A_i$ is a $\prod_{i\in I} S_i$–relative submodule of $\prod_{i\in I} M_i$.

h) Let S be a subring of a ring R and $\{A_i \mid i \in I\}$ an ascending chain of subgroups of an R–module M. Then $\cup_{i\in I} A_i$ is an S–relative submodule of M whenever A_i is an S–relative submodule of M for each $i \in I$.

Definition 3.2. If X is a subset of a module M over a ring R and S is a subring of R, then the intersection of all S–relative submodules of M containing X is called the S–relative submodule generated by X or spanned by X and is denoted by $\langle X\rangle_S$. If X is finite and X generates the S–relative submodule A in M, then A is said to be S–relative finitely generated. If $X = \{a\}$, then $\langle a\rangle_S$ is called the S–relative cyclic submodule generated by a. Finally if $\{B_i \mid i \in I\}$ is a family of S–relative submodules of M, then the S–relative submodule generated by $X = \cup_{i\in I} B_i$ is called the sum of the S–relative submodules B_i. If the index set I is finite, then the sum of $B_1, B_2, \dots, B_n$ is denoted by $B_1 + B_2 + \cdots + B_n$.

<u>Theorem 3.3</u>. Let S be a subring of a ring R, A an R–module, X a subset of A, $\{B_i \mid i \in I\}$ a family of S–relative submodules of A, a an element in A, and $Sa = \{sa \mid s \in S\}$.

1) Sa is an S–relative submodule of A and the map $S \to Sa$ given by $s \vdash sa$ is an S–module epimorphism.
2) The S–relative cyclic submodule C generated by a is $\{sa + na \mid s \in S$ and $n \in \mathbb{Z}$ the ring of integers$\}$. If S has an identity 1_S and $1_S a = a$, then $C = Sa$.
3) The S–relative submodule D generated by X is the set of all elements of the form $\sum_{i=1}^{n} s_i a_i + \sum_{j=1}^{m} n_j b_j$ where n, m are non–negative integers, $n_j \in \mathbb{Z}$, $s_i \in S$ and $a_i, b_j \in X$. If S has an identity 1_S and for each $x \in X$, $1_S x = x$, then $D = SX = \{\sum_{i=1}^{n} s_i a_i \mid s_i \in S,\ a_i \in X,$ and n a non–negative integer$\}$.
4) The sum of the family $\{B_i \mid i \in I\}$ consists of all finite sums $b_{i_1} + b_{i_2} + \cdots + b_{i_n}$ where b_{i_k} is an element of B_{i_k}.

<u>Proof</u>. The proof follows directly from the definition.

<u>Definition 3.3</u>. Let S and T with $S \subseteq T$ be two subrings of a ring R, A an R–module and B a T–module. A group homomorphism $f\colon A \to B$ is said to be an S–relative homomorphism of modules if for all $s \in S$ and $a \in A$, $f(sa) = sf(a)$.

<u>Theorem 3.4</u>. Let S be a subring of a ring R and B an S–relative submodule of a module A over R. Then the quotient group A/B is an S-module with the action of S on A/B given by $s(a + B) = sa + B$ for all $s \in S$ and $a \in A$. The map $\pi_S\colon A \to A/B$ given by $a \vdash a + B$ is an S–relative epimorphism of modules with the kernel B. The map π_S is called the S–relative canonical epimorphism or projection.

<u>Proof</u>. If $a+B = a'+B$, then $a-a' \in B$. Since B is an S–relative submodule of A, then $sa - sa' = s(a-a')$ is an element in B for all s in S. Thus, $sa+B = sa'+B$ which implies that the action of S on A/B is well defined. The remainder of the proof is left to the reader.

<u>Definition 3.4</u>. Let A and B be two R–modules over a ring R and $f\colon A \to B$ a group homomorphism. The set of all r in R such that for each r there exists an element a in A with $f(ra) \neq rf(a)$ is called the rejective set of f in R and is denoted by $R(f, R)$ or $R(f)$ whenever there is no confusion in the context.

Remark. From the above definition, it is clear that $f\colon A \to B$ is an R–module homomorphism if and only if $R(f,R)$ is the empty set. Note that zero is always in $R \setminus R(f)$ since $f(0a) = f(0) = 0 = 0f(a)$. In addition if $f\colon A \to B$ is a group homomorphism of two unitary R–modules A and B, then $f(1_R a) = f(a) = 1_R f(a)$ which implies $1_R \in R \setminus R(f)$.

Theorem 3.5. Assume each of A and B is an R–module over a ring R. If $f\colon A \to B$ is a group homomorphism, then $R \setminus R(f)$ the set theoretic complement of the rejective set of f in R is a subring of R and f is an $R \setminus R(f)$–relative homomorphism of A and B. In addition, $R \setminus R(f)$ is a subfield of R whenever R is a field and A and B are unitary R–modules.

Proof. For any $r, s \in R \setminus R(f)$ and $a \in A$, $f((r-s)a) = f(ra - sa) = f(ra) + f(-sa) = rf(a) - sf(a) = (r-s)f(a)$ which implies $(r-s)$ is in $R \setminus R(f)$. Similarly, $f((rs)a) = f(r(sa)) = rf(sa) = (rs)f(a)$ implies rs is in $R \setminus R(f)$. Now suppose R is a field and r is an arbitrary nonzero element of $R \setminus R(f)$. Thus, for any a in A, $f(a) = f(rr^{-1}a) = rf(r^{-1}a)$ which implies $r^{-1}f(a) = f(r^{-1}a)$.

Corollary 3.2. Let S be a subring of a ring R and $f\colon A \to B$ a group homomorphism of the R–modules A and B. Then f is an $R \setminus R(f)$–relative homomorphism of the R–modules A and B and S is contained in $R \setminus R(f)$ whenever f is an S–relative homomorphism of A and B.

Definition 3.5. Let A and B be two R–modules over a ring R and $f\colon A \to B$ a homomorphism of the groups. The set of all a in A such that for each a there exists an element r in R with $f(ra) \neq rf(a)$ is called the non–absorptive set of f in R and is denoted by $N(f,R)$ or $N(f)$ whenever there is no confusion in the context.

Remark. In the above definition, it is clear that f is an R–module homomorphism of A and B if and only if $N(f)$ is the empty set.

Theorem 3.6. Let A and B be two R–modules over a ring R and $f\colon A \to B$ a group homomorphism. Then $A \setminus N(f)$ the set theoretic complement of the non–absorptive set of f in A is a submodule of A.

Proof. Note that $A \setminus N(f)$ is a non–empty set since it contains the zero element $f(r0) = f(0) = 0 = rf(0)$ for any r in R. For any $r \in R$ and $a, b \in A \setminus N(f)$, $f(r(a-b)) = f(ra-rb) = f(ra)+f(-rb) = rf(a)+rf(-b) = rf(a-b)$ which implies $a-b$ is in $A \setminus N(f)$. Now suppose for some a in $A \setminus N(f)$ there exists an r in R such that $ra \ni A \backslash N(f)$. Then there exists $s \in R$ such that $f(s(ra)) \neq sf(ra) = (sr)f(a)$ which is a contradicition to the choice of a in $A \setminus N(f)$.

In conclusion, it should be noted that the above ideas are new to the author and a search of the literature found no mention of such a concept as presented here. It is entirely possible, however, that a reader might know of a source of similar ideas.

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