SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

21. [1990, 80; 1991, 95] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find distinct positive integers, a, b, c, d such that

$$a+b+c+d+abcd = ab+bc+ca+ad+bd+cd+abc+abd+acd+bcd.$$

Solution by Les Reid, Southwest Missouri State University, Springfield, Missouri.

More generally, we will find all positive integer solutions to

$$a+b+c+d+abcd = abc+abd+acd+bcd+ab+ac+ad+bc+bd+cd.$$

Rewriting as

$$abcd-abc-abd-acd-bcd-ab-ac-ad-bc-cd+a+b+c+d=0$$

and noting that the left-hand side has the opposite parity of $(a-1)(b-1)(c-1)(d-1)$ [since their difference is $2(ab+ac+ad+bc+bd+cd-a-b-c-d)+1$], we see that $(a-1)(b-1)(c-1)(d-1)$ must be odd, hence a, b, c, and d must all be even.

Since our original equation is symmetric in the variables, we may assume without loss of generality that $0<d\le c\le b\le a$.

If $d=2$ and $c=2$, our equation is equivalent to $ab+7a+7b=0$ which has no positive solutions.

If $d=2$ and $c=4$, our equation may be rewritten as $(a-13)(b-13)=171$. Since 171 can be factored as $171\cdot 1$, $57\cdot 3$, $19\cdot 9$, we obtain the solutions $a=184$, $b=14$; $a=70$, $b=16$; $a=32$, $b=22$.

If $d=2$ and $c=6$, our equation may be rewritten as $(3a-19)(3b-19)=373$. Since 373 is prime, the only possible factorization is as $373\cdot 1$, but this yields non-integer values for a and b.

If $d=2$ and $c=8$, our equation becomes $5ab-25a-25b=6$, which is clearly impossible since the left-hand side is not divisible by 5.

If $d = 2$ and $c \geq 10$, then

$$\begin{gathered} abcd - abc - abd - acd - bcd - ab - ac - ad - bc - bd - cd + a + b + c + d = \\ abc - 3ab - 3ac - 3bc - a - b - c + 2 \geq 10ab - 3ab - 3ab - 3ab - a - b - b + 2 = \\ ab - a - 2b + 2 = (a-2)(b-1) > 0 \end{gathered}$$

so there are no further solutions when $d = 2$.

If $d = 4$ and $c = 4$, our equation may be rewritten as $(7a - 23)(7b - 23) = 585$. The only factorization of 585 whose factors are both congruent to $-23 \equiv 5 \mod 7$ is $117 \cdot 5$, which yields $a = 20$, $b = 4$.

If $d = 4$ and $c \geq 6$, then

$$\begin{aligned} abcd - abc - abd - acd - bcd - ab - ac - ad - bc - bd - cd + a + b + c + d \\ &= 3abc - 5ab - 5ac - 5bc - 3a - 3b - 3c + 4 \\ &\geq 18ab - 5ab - 5ab - 5ab - 3a - 3b - 3b + 4 \\ &= 3ab - 3a - 6b + 4 \\ &= 3(a-2)(b-1) - 2 \geq 3 \cdot 4 \cdot 5 - 2 > 0. \end{aligned}$$

If $d \geq 6$, then

$$\begin{gathered} abcd - abc - abd - acd - bcd - ab - ac - ad - bc - bd - cd + a + b + c + d \geq \\ 6abc - abc - abc - abc - abc - ab - ac - ab - bc - ab - bc + a + b + c + 6 = \\ 2abc - 3ab - 2bc - ac + a + b + c + 6 \geq 12ab - 3ab - 2ab - ab + a + b + 6 + 6 = \\ 6ab + a + b + 12 > 0 \end{gathered}$$

so there are no more positive solutions.

Therefore, the only positive solutions of our equation (up to reordering) are

$$\begin{aligned} &a = 184,\ b = 14,\ c = 4,\ d = 2 \\ &a = 70,\ b = 16,\ c = 4,\ d = 2 \\ &a = 32,\ b = 22,\ c = 4,\ d = 2 \\ &a = 20,\ b = 4,\ c = 4,\ d = 4. \end{aligned}$$

133. [2001,206] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \frac{k}{\log\left(1+\frac{1}{k}\right)} < \frac{n^2(n+2)}{2}.$$

Here, log denotes the natural logarithm.

Solution by Joe Howard, Portales, New Mexico and Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. We use the Logarithmic-Arithmetic Mean inequality, which says that for $a > b > 0$,

$$\frac{a-b}{\log a - \log b} < \frac{a+b}{2}.$$

Since

$$\log\left(1+\frac{1}{k}\right) = \log(k+1) - \log k,$$

we have

$$\frac{1}{\log(k+1) - \log k} < \frac{2k+1}{2}.$$

Multiplying by $k > 0$ it follows that

$$\sum_{k=1}^{n} \frac{k}{\log(1+\frac{1}{k})} < \sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{4} = \frac{n(n+1)(4n+5)}{12}.$$

Note that

$$\frac{n(n+1)(4n+5)}{12} \le \frac{n^2(n+2)}{2} \quad \text{for } n \ge 1,$$

since

$$4n^2 + 9n + 5 \le 6n^2 + 12n \text{ if and only if } 5 \le 2n^2 + 3n \quad \text{for } n \ge 1.$$

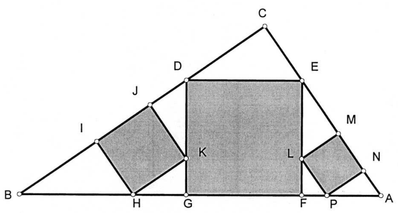
Also solved by Les Reid, Southwest Missouri State University, Springfield, Missouri; J. D. Chow, Edinburg, Texas; Joseph Dence, University of Missouri-St. Louis, St. Louis, Missouri; Craig Haile, College of the Ozarks, Point Lookout, Missouri; Alan H. Rapoport, Santurce, Puerto Rico; Joe Howard, Portales, New Mexico (2 solutions); and the proposer.

Les Reid's solution, in addition to giving the stronger upper bound in the featured solution, gave a lower bound for the sum, i.e.,

$$\frac{n(4n^2+9n+4)}{12} < \sum_{k=1}^{n} \frac{k}{\log\left(1+\frac{1}{k}\right)} \quad \text{for } n \ge 1.$$

134. [2001,206] *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, Tennessee.*

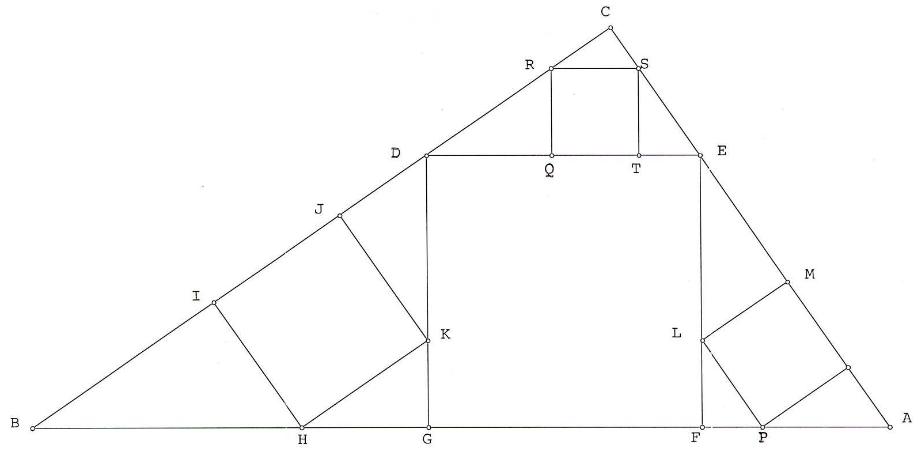
Let square DEFG be inscribed in right triangle ABC, square HIJK be inscribed in triangle GBD, and square LMNP be inscribed in triangle AFE as shown in the figure. Prove or disprove that KG = LF.



Solution I by Clayton W. Dodge, University of Maine, Orono, Maine. Clearly, all the right triangles and their inscribed squares are similar, so that corresponding parts are proportional. In particular, $MN/EA = IJ/BD$. Also, $KG/KH = DG/BD$ and $LF/LP = EF/EA$. Then we have

$$LF = \frac{EF \cdot LP}{EA} = DG \cdot \frac{MN}{EA} = DG \cdot \frac{IJ}{BD} = \frac{DG \cdot KH}{BD} = KG.$$

Solution II by Leon Hall, University of Missouri - Rolla, Rolla, Missouri.



Let $a = BC$, $b = AC$, $c = AB$, $a_1 = CD$, $b_1 = CE$, and $s = FG$. Then from similar triangles,

$$\frac{a_1}{a} = \frac{b_1}{b} = \frac{s}{c}$$

and

$$\frac{KG}{DG} = \frac{KG}{s} = \frac{b_1}{b} \text{ so } KG = \frac{s^2}{c},$$

$$\frac{LF}{EF} = \frac{LF}{s} = \frac{a_1}{a} \text{ so } LF = \frac{s^2}{c},$$

and thus, $KG = LF$.

Leon Hall also had several remarks regarding the problem.

<u>Remark 1</u>. There is a third length also equal to KG and LF. Inscribe square $QRST$ in triangle CDE as shown in the figure. Then

$$\frac{QT}{s} = \frac{s}{c} \text{ so } QT = \frac{s^2}{c}.$$

Remark 2. There ought to be some relationship(s) between the three small squares, and there is. Let $s_1 = HK$, $s_2 = LP$, and $s_3 = QT$. Then, again using similar figures,

$$\frac{s_1}{s} = \frac{s}{b}, \quad \frac{s_2}{s} = \frac{s}{a}, \quad \text{and} \quad \frac{s_3}{s} = \frac{s}{c}.$$

This leads to the relationships

$$bs_1 = as_2 = cs_3 = s^2$$

and

$$\frac{1}{s_1^2} + \frac{1}{s_2^2} = \frac{1}{s_3^2}.$$

Remark 3. Now disregard the three small squares and consider the three triangles CDE, BDG, and AEF, together with square $DEFG$. If the areas of the triangles are denoted T_1, T_2, and T_3 respectively, then some more similar triangle work gives

$$T_1 = \frac{abs^2}{2c^2}, \quad T_2 = \frac{as^2}{2b}, \quad T_3 = \frac{bs^2}{2a}.$$

From these it follows that

$$\frac{s^2}{2} = \sqrt{T_1(T_2 + T_3)},$$

or, in words, half the area of square $DEFG$ is the geometric mean of the area of triangle CDE and the sum of the areas of triangles BDG and AEF.

Remark 4. This problem is similar to one of the problems in the May 1998 *Scientific American* article, "Japanese Temple Geometry", by Tony Rothman. There, squares $LMNP$ and $QRST$ are not drawn, and a new square is inscribed in triangle BHI. Then circles are inscribed in triangles AEF, DJK, and the triangle formed by the new square, square $HIJK$, and segment AB. The radius of the circle inscribed in triangle DJK (the middle-sized one) is the geometric mean of the radii of the other two circles.

Also solved by Joe Howard, Portales, New Mexico; Joseph Dence, University of Missouri - St. Louis, St. Louis, Missouri; Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan; Alan H. Rapoport, Santurce, Puerto Rico (4 solutions); and the proposer.

135. [2001,207] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let $z_0, z_1, \ldots, z_n$ be $n+1$ complex numbers lying in the closed left half plane $\mathrm{Re}(z) \le 0$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{|1-z_k|}{1+|z_k|} \right\}^2 \ge 2^{n-1}.$$

When does equality occur?

Solution by Joseph B. Dence, University of Missouri - St. Louis, St. Louis, Missouri.

Since $|\mathrm{Re}(z_k)| \le |z_k|$, then by hypothesis $-\mathrm{Re}(z_k) \le |z_k|$ and $|1-z_k|^2 = 1 - 2\mathrm{Re}(z_k) + |z_k|^2 \le 1 + 2|z_k| + |z_k|^2 = \{1+|z_k|\}^2$. So for each k

$$0 \le \left\{ \frac{|1-z_k|}{1+|z_k|} \right\}^2 \le 1,$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{|1-z_k|}{1+|z_k|} \right\}^2 \le \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

On the other hand, letting $z_k = r_k e^{i\phi_k}$, we have

$$\left\{ \frac{|1-z_k|}{1+|z_k|} \right\}^2 = \frac{1 + r_k^2 - 2r_k \cos\phi_k}{1 + 2r_k + r_k^2}, \quad \frac{\pi}{2} \le \phi_k \le \frac{3\pi}{2}.$$

Clearly, the left-hand side is minimized at $\phi_k = \frac{\pi}{2}, \frac{3\pi}{2}$ and for some r_k. Let $f(r_k) = (1+r_k^2)/(1+2r_k+r_k^2)$; then $f'(r_k) = 0$ implies $r_k = 1$. Further, $f''(1) = 1/16 > 0$, so $r_k = 1$ corresponds to a minimum. Hence, for each k

$$\left\{ \frac{1 + r_k^2 - 2r_k \cos\phi_k}{1 + 2r_k + r_k^2} \right\} \ge \frac{1 + 1^2 - 0}{1 + 2 + 1^2} = \frac{1}{2},$$

and finally,

$$\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{|1-z_k|}{1+|z_k|}\right\}^2 \geq \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right) = 2^n \cdot \frac{1}{2} = 2^{n-1}.$$

Equality occurs when each $z_k \in \{i, -i\}$.

Also solved by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan and the proposer.

136. [2001,207] *Proposed by Kenneth B. Davenport, 301 Morea Road, Frackville, Pennsylvania.*

Show that if

$$A = \sum_{n=0}^{\infty}\left(\frac{1}{15n+1} - \frac{1}{15n+6}\right), \qquad B = \sum_{n=0}^{\infty}\left(\frac{1}{15n+9} - \frac{1}{15n+14}\right),$$

$$C = \sum_{n=0}^{\infty}\left(\frac{1}{15n+2} - \frac{1}{15n+7}\right), \qquad D = \sum_{n=0}^{\infty}\left(\frac{1}{15n+8} - \frac{1}{15n+13}\right),$$

$$E = \sum_{n=0}^{\infty}\left(\frac{1}{15n+4} - \frac{1}{15n+9}\right), \qquad F = \sum_{n=0}^{\infty}\left(\frac{1}{15n+6} - \frac{1}{15n+11}\right),$$

then $A + B = (C + D) + (E + F)\beta$, where $\beta = 2\cos(2\pi/15)$.

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. I will make use of the following formulae:

$$\Psi(z) - \Psi(1-z) = -\pi\cot\pi z. \tag{1}$$

and

$$\Psi(z+1) = -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right); \quad z \neq -1, -2, -3, \dots, \tag{**}$$

where γ is the Euler constant and

$$\Psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)};$$

Notice that

$$A - -\frac{1}{15}\Psi\left(\frac{1}{15}\right) + \frac{1}{15}\Psi\left(\frac{2}{5}\right) \qquad (*)$$

I used MAPLE to find A but I can also provide an algebraic explanation for the above formula (at the end of the problem). So we get that:

$$A = -\frac{1}{15}\Psi\left(\frac{1}{15}\right) + \frac{1}{15}\Psi\left(\frac{2}{5}\right), \qquad B = -\frac{1}{15}\Psi\left(\frac{3}{5}\right) + \frac{1}{15}\Psi\left(\frac{14}{15}\right),$$

$$C = -\frac{1}{15}\Psi\left(\frac{2}{15}\right) + \frac{1}{15}\Psi\left(\frac{7}{15}\right), \qquad D = -\frac{1}{15}\Psi\left(\frac{8}{15}\right) + \frac{1}{15}\Psi\left(\frac{13}{15}\right),$$

$$E = -\frac{1}{15}\Psi\left(\frac{4}{15}\right) + \frac{1}{15}\Psi\left(\frac{3}{5}\right), \qquad F = -\frac{1}{15}\Psi\left(\frac{2}{5}\right) + \frac{1}{15}\Psi\left(\frac{11}{15}\right),$$

$$A + B = \frac{1}{15}\left[\Psi\left(\frac{2}{5}\right) - \Psi\left(\frac{3}{5}\right)\right] + \frac{1}{15}\left[\Psi\left(\frac{14}{15}\right) - \Psi\left(\frac{1}{15}\right)\right],$$

and according to (1) we get

$$A + B = -\frac{\pi}{15} \cdot \cot\frac{2\pi}{5} + \frac{-\pi}{15}\cot\frac{14\pi}{15}$$

$$= -\frac{\pi}{15} \cdot \left(\cot\frac{2\pi}{5} + \cot\frac{14\pi}{15}\right) = -\frac{\pi}{15}\left(\cot\frac{6\pi}{15} + \cot\frac{14\pi}{15}\right)$$

$$= \frac{2\pi}{5} = \frac{6\pi}{15} = -\frac{\pi}{15} \cdot \frac{\sin\frac{20\pi}{15}}{\sin\frac{6\pi}{15} \cdot \sin\frac{14\pi}{15}} = -\frac{\pi}{15} \cdot \frac{\sin\frac{4\pi}{3}}{\sin\frac{2\pi}{5}\sin\frac{14\pi}{15}}.$$

But

$$\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} \quad \text{and} \quad \sin\frac{14\pi}{15} = \sin\frac{\pi}{15},$$

so we obtain that

$$A + B = \frac{\pi}{15} \cdot \frac{\sin\frac{\pi}{3}}{\sin\frac{2\pi}{5} \cdot \sin\frac{\pi}{15}}. \tag{2}$$

Analogously,

$$C + D = \frac{\pi}{15} \cdot \frac{\sin\frac{\pi}{3}}{\sin\frac{7\pi}{15} \cdot \sin\frac{2\pi}{15}} \quad \text{and} \quad E + F = \frac{\pi}{15} \cdot \frac{\sin\frac{\pi}{3}}{\sin\frac{3\pi}{5} \cdot \sin\frac{4\pi}{15}},$$

so

$$C + D + \beta(E + F) = C + D + 2\cos\frac{2\pi}{15}(E + F)$$

$$= \frac{\pi}{15} \cdot \frac{\sin\frac{\pi}{3}}{\sin\frac{7\pi}{15} \cdot \sin\frac{2\pi}{15}} + 2\cos\frac{2\pi}{15} \cdot \frac{\pi}{15} \cdot \frac{\sin\frac{\pi}{3}}{\sin\frac{3\pi}{5} \cdot \sin\frac{4\pi}{15}}.$$

But

$$\sin\frac{4\pi}{15} = 2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15},$$

so

$$C + D + \beta(E + F) = \frac{\pi}{15} \cdot \sin\frac{\pi}{3}\left[\frac{1}{\sin\frac{7\pi}{15}\sin\frac{2\pi}{15}} + \frac{1}{\sin\frac{3\pi}{15}\sin\frac{2\pi}{15}}\right]$$

$$= \frac{\pi}{15}\frac{\sin\frac{\pi}{3}}{\sin\frac{2\pi}{15}} \cdot \frac{\sin\frac{3\pi}{5} + \sin\frac{7\pi}{15}}{\sin\frac{3\pi}{5}\sin\frac{7\pi}{15}}.$$

But

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2},$$

so

$$\sin\frac{3\pi}{5}+\sin\frac{7\pi}{15}=\sin\frac{9\pi}{15}+\sin\frac{7\pi}{15}=2\sin\frac{8\pi}{15}\cdot\cos\frac{\pi}{15}.$$

Thus,

$$C+D+\beta(E+F)=\frac{\pi}{15}\cdot\frac{\sin\frac{\pi}{3}}{\sin\frac{2\pi}{15}}\cdot\frac{2\sin\frac{8\pi}{15}\cos\frac{\pi}{15}}{\sin\frac{3\pi}{5}\sin\frac{7\pi}{15}}.$$

But

$$\sin\frac{2\pi}{5}=2\sin\frac{\pi}{5}\cos\frac{\pi}{5}\quad;\quad\sin\frac{8\pi}{15}=\sin\frac{7\pi}{15},$$

so

$$C+D+\beta(E+F)=\frac{\pi}{15}\cdot\frac{\sin\frac{\pi}{3}}{\sin\frac{\pi}{15}\sin\frac{3\pi}{5}}. \tag{3}$$

From (2) and (3) and the fact that

$$\sin\frac{3\pi}{5}=\sin\frac{2\pi}{5},$$

the result follows.

To explain (*), we note that

$$A=\sum_{n=0}^{\infty}\left(\frac{1}{15n+1}-\frac{1}{15n+6}\right)=-\frac{1}{15}\Psi\left(\frac{1}{15}\right)+\frac{1}{15}\Psi\left(\frac{2}{5}\right).$$

From first using (**) and later letting $n-1=k$, we observe that

$$\Psi\left(\frac{1}{15}\right)=\Psi\left(1-\frac{14}{15}\right)=-\gamma-\sum_{n=1}^{\infty}\left(\frac{1}{n-\frac{14}{15}}-\frac{1}{n}\right)$$

$$=-\gamma-\sum_{n=1}^{\infty}\left(\frac{15}{15n-14}-\frac{1}{n}\right)=-\gamma-\sum_{n=1}^{\infty}\left(\frac{15}{15(n-1)+1}-\frac{1}{n}\right)$$

$$=-\gamma-\sum_{k=0}^{\infty}\left(\frac{15}{15k+1}-\frac{1}{k+1}\right).$$

Also,

$$\Psi\left(\frac{2}{15}\right) = \Psi\left(\frac{6}{15}\right) = \Psi\left(1 - \frac{9}{15}\right) = -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n - \frac{9}{15}} - \frac{1}{n}\right)$$

$$= -\gamma - \sum_{n\geq 1}\left(\frac{15}{15n-9} - \frac{1}{n}\right) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{15}{15k+6} - \frac{1}{k+1}\right).$$

Observe that

$$15A = 15 \cdot \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{1}{15k+1} - \frac{1}{15k+6}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{15}{15k+1} - \frac{1}{k+1} + \frac{1}{k+1} - \frac{15}{15k+6}\right)$$

$$= \lim_{n\to\infty}\left\{\left[-\gamma - \sum_{k=0}^{n}\left(\frac{15}{15k+6} - \frac{1}{k+1}\right)\right] - \left[-\gamma - \sum_{k=0}^{n}\left(\frac{15}{15k+1} - \frac{1}{k+1}\right)\right]\right\}$$

$$= -\gamma - \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{15}{15k+6} - \frac{1}{k+1}\right) - \left[-\gamma - \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{15}{15k+1} - \frac{1}{k+1}\right)\right]$$

$$= \Psi\left(\frac{2}{5}\right) - \Psi\left(\frac{1}{15}\right).$$

Therefore,

$$15A = \Psi\left(\frac{2}{5}\right) - \Psi\left(\frac{1}{15}\right).$$

We can analogously obtain the values of B, C, D, E, and F.

Also solved by the proposer.