A NONHOMOLOGICAL PROOF OF SEMIPERFECTNESS IN MATRIX RINGS

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Introduction. Let R be an associative ring with unit. An element e of R is said to be an *idempotent* if $e^2 = e$. Two idempotents e, f of R are said to be *orthogonal* if $ef = fe = 0$. A nonzero idempotent e of R is said to be *primitive* if it cannot be written as the sum of two nonzero orthogonal idempotents. If e is an idempotent of R such that eRe is a local ring, that is eRe has exactly one maximal ideal, then e is said to be *local*. It is known (see [2] for example) that every local idempotent is primitive. However, the converse is not necessarily true. For example, 1 is a primitive but not local idempotent of $\mathbb{Z}$. For an ideal I of R, we say that idempotents of R/I can be *lifted* to R if for every idempotent $u + I \in R/I$, there exists an idempotent $e^2 = e \in R$ such that $e - u \in I$.

Denote the Jacobson radical of R by $J(R)$ and the ring of $n \times n$ matrices over R by $M_n(R)$. R is said to be *semiperfect* if $R/J(R)$ is Artinian and idempotents of $R/J(R)$ can be lifted to R. It has been shown by Kaye [1] via the Morita Duality Theorem that R is semiperfect if and only if $M_n(R)$ is semiperfect. The purpose of this paper is to give a nonhomological proof of this result.

All rings considered in this paper are assumed to be associative with unit.

1. Some Preliminaries. We first state the following result by B. J. Mueller [3] on conditions on a ring which are equivalent to being semiperfect.

Theorem 1.1. Let R be a ring. The following conditions are equivalent:

(i) R is semiperfect;
(ii) Every primitive idempotent of R is local and there is no infinite set of orthogonal idempotents in R;
(iii) The unit $1 \in R$ is the finite sum of some orthogonal local idempotents.

In what follows, for $i, j = 1, \ldots, n$, we let $E_{ij} = (e_{rs})$ denote the $n \times n$ matrix over R such that

$$e_{rs} = \begin{cases} 1 & \text{if } (r, s) = (i, j) \\ 0 & \text{if } (r, s) \neq (i, j) \end{cases}, \quad r, s = 1, \ldots, n.$$

Proposition 1.2. Let R be a ring. If e is a primitive idempotent of R, then eE_{tt} is a primitive idempotent of $M_n(R)$ for $t = 1, \ldots, n$.

Proof. Clearly, eE_{tt} is an idempotent of $M_n(R)$. Suppose that eE_{tt} is not primitive. Then we may write

$$eE_{tt} = X + Y$$

for some nonzero orthogonal idempotents $X = (x_{ij})$, $Y = (y_{ij}) \in M_n(R)$. We next note the following properties:

(i) $e = x_{tt} + y_{tt}$;
(ii) $x_{ij} = -y_{ij}$ for all i, j, $(i, j) \neq (t, t)$;
(iii) $X^2 = X \Rightarrow \sum_{k=1}^{n} x_{ik}x_{kj} = x_{ij}$ for all i, j;
(iv) $Y^2 = Y \Rightarrow \sum_{k=1}^{n} y_{ik}y_{kj} = y_{ij}$ for all i, j;
(v) $XY = 0 \Rightarrow \sum_{k=1}^{n} x_{ik}y_{kj} = 0$ for all i, j;
(vi) $YX = 0 \Rightarrow \sum_{k=1}^{n} y_{ik}x_{kj} = 0$ for all i, j.

We now show that $x_{ij} = 0$ for all $(i, j) \neq (t, t)$. In order to do this, we shall show that

(I) $x_{ij} = 0$ for all $i = 1, \ldots, n$; $j = 1, \ldots, t-1, t+1, \ldots, n$, and
(II) $x_{ij} = 0$ for all $i \neq t$, $j = t$.

We first show (I). Let $j \in \{1, \ldots, t-1, t+1, \ldots, n\}$. Then $(k, j) \neq (t, t)$ for any $k = 1, \ldots, n$. By (ii) we have that $x_{kj} = -y_{kj}$ for every $k = 1, \ldots, n$. It follows that

$$\begin{aligned} x_{ij} &= \sum_{k=1}^{n} x_{ik}x_{kj} \quad \text{(by (iii))} \\ &= \sum_{k=1}^{n} x_{ik}(-y_{kj}) \\ &= -\sum_{k=1}^{n} x_{ik}y_{kj} = 0 \quad \text{(by (v))} \end{aligned}$$

for every $i = 1, \ldots, n$. Hence, (I) is proven. Next we shall show (II). Let $i \in \{1, \ldots, t-1, t+1, \ldots, n\}$. Then $(i, k) \neq (t, t)$ and it follows from (ii) that

$x_{ik} = -y_{ik}$ for every $k = 1, \ldots, n$. Therefore,

$$\begin{aligned} x_{it} &= \sum_{k=1}^{n} x_{ik}x_{kt} \quad \text{(by (iii))} \\ &= \sum_{k=1}^{n} (-y_{ik})x_{kt} \\ &= -\sum_{k=1}^{n} y_{ik}x_{kt} = 0 \quad \text{(by (vi))}. \end{aligned}$$

Hence, (II) is proven. Thus, we have shown that $x_{ij} = 0$ for all $(i, j) \neq (t, t)$. By (ii), it then follows that $y_{ij} = 0$ for all $(i, j) \neq (t, t)$.

Now by (iii),

$$x_{tt}^2 + \sum_{\substack{k=1 \\ k \neq t}}^{n} x_{tk}x_{kt} = x_{tt}.$$

But since $x_{tk} = 0$ for all $k \neq t$, we have that $x_{tt}^2 = x_{tt}$. Next, from (v) we have

$$x_{tt}y_{tt} + \sum_{\substack{k=1 \\ k \neq t}}^{n} x_{tk}y_{kt} = 0.$$

But since $x_{tk} = 0$ for all $k \neq t$, we have that $x_{tt}y_{tt} = 0$. Note that since $y_{ij} = 0$ for all $(i, j) \neq (t, t)$, it can be similarly shown that $y_{tt}^2 = y_{tt}$ and $y_{tt}x_{tt} = 0$. Also, note that $x_{tt} \neq 0$ and $y_{tt} \neq 0$. Indeed, if $x_{tt} = 0$, then $X = (x_{ij}) = 0$; a contradiction. Similarly, we would have a contradiction if $y_{tt} = 0$. Thus, we have shown that $e = x_{tt} + y_{tt}$ where $0 \neq x_{tt} = x_{tt}^2$, $0 \neq y_{tt} = y_{tt}^2$ and $x_{tt}y_{tt} = y_{tt}x_{tt} = 0$. This contradicts the fact that e is a primitive idempotent of R. Hence, eE_{tt} must be a primitive idempotent of $M_n(R)$.

Lemma 1.3. Let R be a ring and let e be a nonzero idempotent of R. Then $(eE_{tt})M_n(R)(eE_{tt}) \cong eRe$ for every $t = 1, \ldots, n$.

Proof. Define $\theta\colon (eE_{tt})M_n(R)(eE_{tt}) \to eRe$ as follows:

$$\theta((eE_{tt})(x_{ij})(eE_{tt})) = ex_{tt}e, \quad (x_{ij}) \in M_n(R).$$

By routine verification, θ is a ring isomorphism.

2. The Main Result. The main result in this paper has been obtained by Kaye [1] and is as follows:

Theorem 2.1. A ring R is semiperfect if and only if $M_n(R)$ is semiperfect.

We give here a different proof of this result using the preliminaries in the preceding section.

Proof of Theorem 2.1. Suppose that R is semiperfect. By condition (iii) of Theorem 1.1, there exists a finite set of orthogonal local idempotents $\{e_1, \ldots, e_m\}$ in R such that

$$1 = e_1 + \cdots + e_m.$$

Let I_n denote the identity matrix of $M_n(R)$. Clearly

$$\begin{aligned} I_n = \quad & e_1E_{11} + \cdots + e_mE_{11} \\ & + e_1E_{22} + \cdots + e_mE_{22} + \ldots \\ & + e_1E_{nn} + \cdots + e_mE_{nn}. \end{aligned}$$

We note that $(e_iE_{tt})^2 = e_iE_{tt}$ and $(e_iE_{tt})(e_jE_{ss}) = 0$ if $i \neq j$ or $s \neq t$ where $i, j = 1, \ldots, m$; $s, t = 1, \ldots, n$. We now show that each e_iE_{tt} is local. Since $e_iRe_i \cong (e_iE_{tt})M_n(R)(e_iE_{tt})$ (by Lemma 1.3) and e_iRe_i is local, so is $(e_iE_{tt})M_n(R)(e_iE_{tt})$. It follows that each e_iE_{tt} is local. Thus, we have shown that the identity element of $M_n(R)$ is the finite sum of some orthogonal local idempotents. Hence, by condition (iii) of Theorem 1.1, $M_n(R)$ is semiperfect.

Now suppose that $M_n(R)$ is semiperfect. Note that if e_i and e_j are orthogonal idempotents of R, then e_iE_{11} and e_jE_{11} are orthogonal idempotents of $M_n(R)$. Then since there is no infinite set of orthogonal idempotents in $M_n(R)$ (by (ii) of Theorem 1.1), it follows that there is also no infinite set of orthogonal idempotents in R.

Next we show that every primitive idempotent of R is local. Let e be a primitive idempotent of R. From Proposition 1.2 we have that eE_{11} is a primitive idempotent of $M_n(R)$. Since $M_n(R)$ is semiperfect, so eE_{11} is local and therefore, $(eE_{11})M_n(R)(eE_{11})$ is a local ring. From Lemma 1.3 we have that $eRe \cong (eE_{11})M_n(R)(eE_{11})$ is local. Therefore e is a local idempotent. Thus, we have shown that R satisfies condition (ii) of Theorem 1.1. Hence, R is semiperfect.

References

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