# SOLUTIONS OF TWO PROBLEMS OF D. Ž. DJOKOVIĆ

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**Abstract.** In this paper, solutions of two old problems of D. Ž. Djoković are given. One solution of the first problem is given by O. P. Lossers [2] in implicit form and the other solution by another method is given in explicit form. The second problem is solved here for the first time.

**1. The First Problem.** D. Ž. Djoković formulated the following problem in [1].

Find the eigenvalues and eigenvectors of the two–diagonal matrix $A = (a_{ij})$, where $a_{ij} = 0$ if $\mid i - j \mid\neq 1$ and $a_{i,i+1} = a_{n+2-i,n+1-i} = i$, $(1 \leq i \leq n)$.

Comment on the Problem. O. P. Lossers [2] has proven that if $\lambda$ is an eigenvalue and $(q_1, \cdots, q_{n+1})$ is the corresponding eigenvector, then the possible values of $\lambda$ are $n - 2k$ for $(0 \leq k \leq n)$ and the coordinates $q_i$ of the corresponding eigenvector for $\lambda = n - 2k$ are given by the coefficients of the polynomial $Q(x) = q_1 + q_2x + \cdots + q_{n+1}x^n$, where $Q(x) = (1 + x)^{n-k}(1 - x)^k$, $(0 \leq k \leq n)$ is a generating polynomial. Since he found n+1 eigenvalues which have multiplicity 1 and their corresponding eigenvectors, these are the required eigenvalues and eigenvectors.

Apart from the solution [2], we give another explicit solution of the problem by a different method. We obtain our solution without any information about the Lossers' solution.

D. S. Mitrinović [3] cites this problem two times as an unsolved problem (after publishing the solution [2]) in order to emphasize the difficulty of the problem. D. Ž. Djoković probably formulated this problem based on Problem 2.29 in the book by D. S. Mitrinović and R. B. Potts [4]. The determinant of the matrix $A$ is well–known and found in [5, 6].

Solution of the problem. The characteristic polynomial of $A$ is

$$D_{n+1}(-\lambda) = \det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ n & -\lambda & 2 & \cdots & 0 & 0 & 0 \\ 0 & n-1 & -\lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -\lambda & n \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}.$$

If to each row of the determinant we add all the following rows, we obtain

$$D_{n+1}(-\lambda) = \begin{vmatrix} n-\lambda & n-\lambda & n-\lambda & \cdots & n-\lambda & n-\lambda & n-\lambda \\ n & n-\lambda-1 & n-\lambda & \cdots & n-\lambda & n-\lambda & n-\lambda \\ 0 & n-1 & n-\lambda-2 & \cdots & n-\lambda & n-\lambda & n-\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2-\lambda & n-\lambda & n-\lambda \\ 0 & 0 & 0 & \cdots & 2 & 1-\lambda & n-\lambda \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}.$$

If from each column we subtract the previous column, it follows that

$$D_{n+1}(-\lambda) = (n-\lambda) \begin{vmatrix} -\lambda-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ n-1 & -\lambda-1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & n-2 & -\lambda-1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda-1 & n-2 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -\lambda-1 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda-1 \end{vmatrix},$$

which gives the recurrence formula

$$D_{n+1}(-\lambda) = (n-\lambda) D_n(-\lambda-1).$$

Continuing this process, we obtain

$$
\begin{aligned}
&D_{n+1}(-\lambda) = (n-\lambda)(n-\lambda-2)D_{n-1}(-\lambda-2)\\
&= (n-\lambda)(n-\lambda-2)(n-\lambda-4)D_{n-2}(-\lambda-3)\\
&= \cdots = (n-\lambda)(n-\lambda-2)(n-\lambda-4)\cdots[n-\lambda-2(n-1)]D_1(-\lambda-n)\\
&= (n-\lambda)(n-\lambda-2)(n-\lambda-4)\cdots[n-\lambda-2(n-1)](-\lambda-n)\\
&= \prod_{k=1}^{n+1}[n-\lambda-2(k-1)].
\end{aligned}
$$

Hence, the eigenvalues are

$$\lambda_k = n-2(k-1), \qquad (1 \le k \le n+1).$$

The eigenvectors $\mathbf{X_k}(1 \le k \le n+1)$ of $A$, are determined from the matrix equation

$$(A-\lambda_k I)\mathbf{X_k} = O,$$

where $\mathbf{X_k} = [x_{1k}, x_{2k}, \dots, x_{n+1,k}]^T$ is a non–zero vector determined up to a scalar multiplier. In our case we obtain the following system

$$
\begin{aligned}
&-\lambda_k x_{1k} + x_{2k} = 0\\
&n x_{1k} - \lambda_k x_{2k} + 2x_{3k} = 0\\
&(n-1)x_{2k} - \lambda_k x_{3k} + 3x_{4k} = 0\\
&\qquad\qquad\vdots\\
&3x_{n-2,k} - \lambda_k x_{n-1,k} + (n-1)x_{nk} = 0\\
&2x_{n-1,k} - \lambda_k x_{nk} + n x_{n+1,k} = 0\\
&x_{nk} - \lambda_k x_{n+1,k} = 0,
\end{aligned}
$$

i.e.,

$$(n+2-i)x_{i-1,k} - \lambda_k x_{ik} + i x_{i+1,k} = 0, \qquad (1 \le i,k \le n+1) \tag{1.1}$$

where we have put $x_{0k} = x_{n+2,k} = 0$. We will prove that

$$x_{ik} = \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{k-1}{i-1-j} \binom{n+1-k}{j}, \qquad (1 \le i, k \le n+1). \tag{1.2}$$

If $k = n+1$, from (1.1) we obtain

$$(n+2-i)x_{i-1,n+1} + nx_{i,n+1} + ix_{i+1,n+1} = 0. \tag{1.3}$$

It is easy to verify that $y_i = \binom{n}{i-1}$ satisfies

$$(n+2-i)y_{i-1} - ny_i + iy_{i+1} = 0.$$

Hence, $x_{i,n+1} = (-1)^{i-1} y_i = (-1)^{i-1}\binom{n}{i-1}$ is a solution of (1.3), and (1.2) is satisfied for $k = n+1$.

The eigenvectors for $k = n, n-1, \ldots, 1$, can be determined as follows. By adding the first $i$ $(1 \le i \le n+1)$ equations in (1.1), we obtain

$$(2k-2)(x_{1k} + \cdots + x_{i-1,k}) + x_{ik}(i + 2k - n - 3) + ix_{i+1,k} = 0.$$

Let $A_s = x_{1k} + \ldots + x_{sk}$ $(1 \le s \le n+1)$. Then

$$\begin{aligned} (2k-2)A_{i-1} + (A_i - A_{i-1})(i + 2k - n - 3) + i(A_{i+1} - A_i) = 0, \\ (n+1-i)A_{i-1} - (n+3-2k)A_i + iA_{i+1} = 0, \\ (n'+2-i)A_{i-1} - (n'+2-2k')A_i + iA_{i+1} = 0, \end{aligned} \tag{1.4}$$

where $n' = n-1$ and $k' = k-1$. Now comparing (1.1) and (1.4) we note that the vector $[A_1, A_2, \ldots, A_{n+1}]^T$ satisfies a system of the form (1.1) where $n$ and $k$ are decreased for 1. This fact enables us to determine the eigenvectors for $k = n, n-1, \ldots, 1$. We do this by using the following combinatorial identity.

$$\sum_{j=0}^{r} (-1)^j \binom{t}{j} = (-1)^r \binom{t-1}{r}, \tag{1.5}$$

which is easy to prove by induction of $r$. Hence, (1.2) can be obtained. Indeed, (1.2) can be proven by regressive induction of $k$, using the identity (1.5). This completes the solution of the problem. Finally, we note that the eigenvector for $k=1$ is

$$\left[\binom{n}{0},\binom{n}{1},\cdots,\binom{n}{n}\right]^T.$$

**2. The Second Problem.** In [7, 8] the following problem is stated.

Prove the identity

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = \sum_{k=n-[n/2]}^{n} (-1)^{n-k}\binom{k}{n-k}(2x)^{2k-n}, \tag{2.1}$$

where $P_n(x)$ is Legendre's polynomial.

<u>First Solution</u>. The above may be rewritten in the following form,

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = \sum_{k=0}^{[n/2]} (-1)^k\binom{n-k}{k}(2x)^{n-2k} = C_n^1(x), \tag{2.2}$$

where $C_n^1(x)$ is Gegenbauer's polynomial.

By squaring

$$(1-2xt+t^2)^{-1/2} = \sum_{\nu=0}^{\infty} P_\nu(x)t^\nu,$$

we obtain

$$(1-2xt+t^2)^{-1} = \sum_{\nu=0}^{\infty} a_\nu(x)t^\nu, \qquad a_\nu(x) = \sum_{k=0}^{\nu} P_k(x)P_{\nu-k}(x). \tag{2.3}$$

Next, differentiating (2.3) $n$-times with respect to $t$, we obtain

$$\frac{\partial^n[(1-2xt+t^2)^{-1}]}{\partial t^n}\,|_{t=0}= n!C_n^1(x) = n!\sum_{k=0}^{n} P_k(x)P_{n-k}(x). \tag{2.4}$$

The proof follows immediately from (2.2) and (2.4).
<u>Second Solution</u>. Using

$$P_k(x)=\sum_{i=0}^{[k/2]}(-1)^i\frac{1}{2^k}\binom{2k-2i}{k-i}\binom{k-i}{i}x^{k-2i}$$

it is sufficient to prove that the coefficients of $x^i$ of the left and the right side of (2.1) are equal. By convention, all indices are non–negative integers, such as the indices of the binomial coefficient $\binom{n}{k}$, $0\le k\le n$, unless otherwise specified. According to this convention, we must prove that

$$\sum_{k}\sum_{i+j=s}(-1)^{i+j}\frac{1}{2^n}\binom{2k-2i}{k-i}\binom{k-i}{i}\binom{2n-2k-2j}{n-k-j}\binom{n-k-j}{j}$$
$$=(-1)^s\binom{n-s}{s}2^{n-2s}$$

for $s\in\{0,1,\cdots,[n/2]\}$, i.e.,

$$\sum_{k}\sum_{i}\frac{1}{2^n}\binom{2k-2i}{k-i}\binom{k-i}{i}\binom{2n-2k-2s+2i}{n-k-s+i}\binom{n-k-s+i}{s-i}$$
$$=\binom{n-s}{s}2^{n-2s}. \tag{2.5}$$

In order to transform the left side L of (2.5) into the right side R, we will use the following two identities.

$$\sum_{i+j=s} \binom{a}{i}\binom{b}{j} = \binom{a+b}{s} \tag{2.6}$$

and

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n. \tag{2.7}$$

Identity (2.6) is well–known. In order to prove (2.7), we note that

$$\frac{1}{2^k}\binom{2k}{k} = \binom{k-\frac{1}{2}}{k},$$

and (2.4) is equivalent to

$$\sum_{k=0}^{n} \binom{k-\frac{1}{2}}{k}\binom{n-k-\frac{1}{2}}{n-k} = 1. \tag{2.8}$$

Consider the function $(1-x)^{-\frac{1}{2}}$ for $|x| < 1$. By squaring

$$(1-x)^{-\frac{1}{2}} = 1 - \frac{\frac{-1}{2}}{1!}x + \frac{\frac{-1}{2}\frac{-3}{2}}{2!}x^2 - \frac{\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}}{3!}x^3 + \cdots$$

$$= 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{3}{2}}{2}x^2 + \binom{\frac{5}{2}}{3}x^3 + \cdots ,$$

and comparing the coefficients of $x^n$ in

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

and

$$\left[1+\binom{\frac{1}{2}}{1}x+\binom{\frac{3}{2}}{2}x^2+\binom{\frac{5}{2}}{3}x^3+\cdots\right]\left[1+\binom{\frac{1}{2}}{1}x+\binom{\frac{3}{2}}{2}x^2+\binom{\frac{5}{2}}{3}x^3+\cdots\right]$$

we obtain (2.8).

Replacing $k$ in $k \geq i$ by the non–negative integer variable $t = k - i$ and using (2.6) and (2.7), we obtain

$$\begin{aligned}
L &= \sum_t \sum_i \frac{1}{2^n}\binom{2t}{t}\binom{t}{i}\binom{2n-2s-2t}{n-s-t}\binom{n-s-t}{s-i} \\
&= \sum_t \frac{1}{2^n}\left[\sum_i \binom{t}{i}\binom{n-s-t}{s-i}\right]\binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \sum_t \frac{1}{2^n}\binom{n-s}{s}\binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \frac{1}{2^n}\binom{n-s}{s}\sum_t \binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \frac{1}{2^n}\binom{n-s}{s}4^{n-s} = \binom{n-s}{s}2^{n-2s} = R.
\end{aligned}$$

*References*


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