SOLUTIONS OF TWO PROBLEMS OF D. Ž. DJOKOVIĆ

Ice B. Risteski and Kostadin G. Trencevski

Abstract. In this paper, solutions of two old problems of D. Ž. Djoković are given. One solution of the first problem is given by O. P. Lossers [2] in implicit form and the other solution by another method is given in explicit form. The second problem is solved here for the first time.

1. The First Problem. D. Ž. Djoković formulated the following problem in [1].

Find the eigenvalues and eigenvectors of the two-diagonal matrix $A = (a_{ij})$, where $a_{ij} = 0$ if $| i - j | \neq 1$ and $a_{i,i+1} = a_{n+2-i,n+1-i} = i$, $(1 \leq i \leq n)$.

Comment on the Problem. O. P. Lossers [2] has proven that if λ is an eigenvalue and $(q_1, \cdots, q_{n+1})$ is the corresponding eigenvector, then the possible values of λ are $n - 2k$ for $(0 \leq k \leq n)$ and the coordinates q_i of the corresponding eigenvector for $\lambda = n - 2k$ are given by the coefficients of the polynomial $Q(x) = q_1 + q_2 x + \cdots + q_{n+1} x^n$, where $Q(x) = (1 + x)^{n-k}(1 - x)^k$, $(0 \leq k \leq n)$ is a generating polynomial. Since he found n+1 eigenvalues which have multiplicity 1 and their corresponding eigenvectors, these are the required eigenvalues and eigenvectors.

Apart from the solution [2], we give another explicit solution of the problem by a different method. We obtain our solution without any information about the Lossers' solution.

D. S. Mitrinović [3] cites this problem two times as an unsolved problem (after publishing the solution [2]) in order to emphasize the difficulty of the problem. D. Ž. Djoković probably formulated this problem based on Problem 2.29 in the book by D. S. Mitrinović and R. B. Potts [4]. The determinant of the matrix A is well-known and found in [5, 6].

Solution of the problem. The characteristic polynomial of A is

$$D_{n+1}(-\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ n & -\lambda & 2 & \cdots & 0 & 0 & 0 \\ 0 & n-1 & -\lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -\lambda & n \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}.$$

If to each row of the determinant we add all the following rows, we obtain

$$D_{n+1}(-\lambda) = \begin{vmatrix} n-\lambda & n-\lambda & n-\lambda & \cdots & n-\lambda & n-\lambda & n-\lambda \\ n & n-\lambda-1 & n-\lambda & \cdots & n-\lambda & n-\lambda & n-\lambda \\ 0 & n-1 & n-\lambda-2 & \cdots & n-\lambda & n-\lambda & n-\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2-\lambda & n-\lambda & n-\lambda \\ 0 & 0 & 0 & \cdots & 2 & 1-\lambda & n-\lambda \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{vmatrix}.$$

If from each column we subtract the previous column, it follows that

$$D_{n+1}(-\lambda) = (n-\lambda) \begin{vmatrix} -\lambda-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ n-1 & -\lambda-1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & n-2 & -\lambda-1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda-1 & n-2 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -\lambda-1 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda-1 \end{vmatrix},$$

which gives the recurrence formula

$$D_{n+1}(-\lambda) = (n-\lambda) D_n(-\lambda-1).$$

Continuing this process, we obtain

$$\begin{aligned}
D_{n+1}(-\lambda) &= (n-\lambda)(n-\lambda-2)D_{n-1}(-\lambda-2)\\
&= (n-\lambda)(n-\lambda-2)(n-\lambda-4)D_{n-2}(-\lambda-3)\\
&= \cdots = (n-\lambda)(n-\lambda-2)(n-\lambda-4)\cdots[n-\lambda-2(n-1)]D_1(-\lambda-n)\\
&= (n-\lambda)(n-\lambda-2)(n-\lambda-4)\cdots[n-\lambda-2(n-1)](-\lambda-n)\\
&= \prod_{k=1}^{n+1}[n-\lambda-2(k-1)].
\end{aligned}$$

Hence, the eigenvalues are

$$\lambda_k = n-2(k-1), \qquad (1 \le k \le n+1).$$

The eigenvectors $\mathbf{X_k}(1 \le k \le n+1)$ of A, are determined from the matrix equation

$$(A-\lambda_k I)\mathbf{X_k} = O,$$

where $\mathbf{X_k} = [x_{1k}, x_{2k}, \ldots, x_{n+1,k}]^T$ is a non-zero vector determined up to a scalar multiplier. In our case we obtain the following system

$$\begin{aligned}
&-\lambda_k x_{1k} + x_{2k} = 0\\
&n x_{1k} - \lambda_k x_{2k} + 2x_{3k} = 0\\
&(n-1)x_{2k} - \lambda_k x_{3k} + 3x_{4k} = 0\\
&\qquad\vdots\\
&3x_{n-2,k} - \lambda_k x_{n-1,k} + (n-1)x_{nk} = 0\\
&2x_{n-1,k} - \lambda_k x_{nk} + n x_{n+1,k} = 0\\
&x_{nk} - \lambda_k x_{n+1,k} = 0,
\end{aligned}$$

i.e.,

$$(n+2-i)x_{i-1,k} - \lambda_k x_{ik} + i x_{i+1,k} = 0, \qquad (1 \le i, k \le n+1) \tag{1.1}$$

where we have put $x_{0k} = x_{n+2,k} = 0$. We will prove that

$$x_{ik} = \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{k-1}{i-1-j} \binom{n+1-k}{j}, \qquad (1 \le i, k \le n+1). \tag{1.2}$$

If $k = n+1$, from (1.1) we obtain

$$(n+2-i)x_{i-1,n+1} + nx_{i,n+1} + ix_{i+1,n+1} = 0. \tag{1.3}$$

It is easy to verify that $y_i = \binom{n}{i-1}$ satisfies

$$(n+2-i)y_{i-1} - ny_i + iy_{i+1} = 0.$$

Hence, $x_{i,n+1} = (-1)^{i-1} y_i = (-1)^{i-1}\binom{n}{i-1}$ is a solution of (1.3), and (1.2) is satisfied for $k = n+1$.

The eigenvectors for $k = n, n-1, \dots, 1$, can be determined as follows. By adding the first i $(1 \le i \le n+1)$ equations in (1.1), we obtain

$$(2k-2)(x_{1k} + \cdots + x_{i-1,k}) + x_{ik}(i + 2k - n - 3) + ix_{i+1,k} = 0.$$

Let $A_s = x_{1k} + \ldots + x_{sk}$ $(1 \le s \le n+1)$. Then

$$\begin{gathered}(2k-2)A_{i-1} + (A_i - A_{i-1})(i + 2k - n - 3) + i(A_{i+1} - A_i) = 0,\\ (n+1-i)A_{i-1} - (n+3-2k)A_i + iA_{i+1} = 0,\\ (n'+2-i)A_{i-1} - (n'+2-2k')A_i + iA_{i+1} = 0, \end{gathered} \tag{1.4}$$

where $n' = n-1$ and $k' = k-1$. Now comparing (1.1) and (1.4) we note that the vector $[A_1, A_2, \dots, A_{n+1}]^T$ satisfies a system of the form (1.1) where n and k are decreased for 1. This fact enables us to determine the eigenvectors for $k = n, n-1, \dots, 1$. We do this by using the following combinatorial identity.

$$\sum_{j=0}^{r} (-1)^j \binom{t}{j} = (-1)^r \binom{t-1}{r}, \tag{1.5}$$

which is easy to prove by induction of r. Hence, (1.2) can be obtained. Indeed, (1.2) can be proven by regressive induction of k, using the identity (1.5). This completes the solution of the problem. Finally, we note that the eigenvector for $k = 1$ is

$$\left[\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right]^T.$$

2. The Second Problem. In [7, 8] the following problem is stated.

Prove the identity

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = \sum_{k=n-[n/2]}^{n} (-1)^{n-k} \binom{k}{n-k} (2x)^{2k-n}, \tag{2.1}$$

where $P_n(x)$ is Legendre's polynomial.

First Solution. The above may be rewritten in the following form,

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k} = C_n^1(x), \tag{2.2}$$

where $C_n^1(x)$ is Gegenbauer's polynomial.

By squaring

$$(1 - 2xt + t^2)^{-1/2} = \sum_{\nu=0}^{\infty} P_\nu(x)t^\nu,$$

we obtain

$$(1 - 2xt + t^2)^{-1} = \sum_{\nu=0}^{\infty} a_\nu(x)t^\nu, \qquad a_\nu(x) = \sum_{k=0}^{\nu} P_k(x)P_{\nu-k}(x). \tag{2.3}$$

Next, differentiating (2.3) n-times with respect to t, we obtain

$$\frac{\partial^n[(1-2xt+t^2)^{-1}]}{\partial t^n}\,|_{t=0}= n!C_n^1(x) = n!\sum_{k=0}^{n} P_k(x)P_{n-k}(x). \tag{2.4}$$

The proof follows immediately from (2.2) and (2.4).
Second Solution. Using

$$P_k(x) = \sum_{i=0}^{[k/2]} (-1)^i \frac{1}{2^k}\binom{2k-2i}{k-i}\binom{k-i}{i}x^{k-2i}$$

it is sufficient to prove that the coefficients of x^i of the left and the right side of (2.1) are equal. By convention, all indices are non-negative integers, such as the indices of the binomial coefficient $\binom{n}{k}$, $0 \le k \le n$, unless otherwise specified. According to this convention, we must prove that

$$\sum_k \sum_{i+j=s} (-1)^{i+j}\frac{1}{2^n}\binom{2k-2i}{k-i}\binom{k-i}{i}\binom{2n-2k-2j}{n-k-j}\binom{n-k-j}{j}$$

$$= (-1)^s\binom{n-s}{s}2^{n-2s}$$

for $s \in \{0, 1, \cdots, [n/2]\}$, i.e.,

$$\sum_k \sum_i \frac{1}{2^n}\binom{2k-2i}{k-i}\binom{k-i}{i}\binom{2n-2k-2s+2i}{n-k-s+i}\binom{n-k-s+i}{s-i}$$

$$= \binom{n-s}{s}2^{n-2s}. \tag{2.5}$$

In order to transform the left side L of (2.5) into the right side R, we will use the following two identities.

$$\sum_{i+j=s} \binom{a}{i}\binom{b}{j} = \binom{a+b}{s} \tag{2.6}$$

and

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n. \tag{2.7}$$

Identity (2.6) is well-known. In order to prove (2.7), we note that

$$\frac{1}{2^k}\binom{2k}{k} = \binom{k-\frac{1}{2}}{k},$$

and (2.4) is equivalent to

$$\sum_{k=0}^{n} \binom{k-\frac{1}{2}}{k}\binom{n-k-\frac{1}{2}}{n-k} = 1. \tag{2.8}$$

Consider the function $(1-x)^{-\frac{1}{2}}$ for $|x|<1$. By squaring

$$(1-x)^{-\frac{1}{2}} = 1 - \frac{\frac{-1}{2}}{1!}x + \frac{\frac{-1}{2}\frac{-3}{2}}{2!}x^2 - \frac{\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}}{3!}x^3 + \cdots$$

$$= 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{3}{2}}{2}x^2 + \binom{\frac{5}{2}}{3}x^3 + \cdots,$$

and comparing the coefficients of x^n in

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

and

$$\left[1+\binom{\frac{1}{2}}{1}x+\binom{\frac{3}{2}}{2}x^2+\binom{\frac{5}{2}}{3}x^3+\cdots\right]\left[1+\binom{\frac{1}{2}}{1}x+\binom{\frac{3}{2}}{2}x^2+\binom{\frac{5}{2}}{3}x^3+\cdots\right]$$

we obtain (2.8).

Replacing k in $k \geq i$ by the non-negative integer variable $t = k - i$ and using (2.6) and (2.7), we obtain

$$\begin{aligned}
L &= \sum_t \sum_i \frac{1}{2^n}\binom{2t}{t}\binom{t}{i}\binom{2n-2s-2t}{n-s-t}\binom{n-s-t}{s-i} \\
&= \sum_t \frac{1}{2^n}\left[\sum_i \binom{t}{i}\binom{n-s-t}{s-i}\right]\binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \sum_t \frac{1}{2^n}\binom{n-s}{s}\binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \frac{1}{2^n}\binom{n-s}{s}\sum_t \binom{2t}{t}\binom{2n-2s-2t}{n-s-t} \\
&= \frac{1}{2^n}\binom{n-s}{s}4^{n-s} = \binom{n-s}{s}2^{n-2s} = R.
\end{aligned}$$

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Ice B. Risteski
2 Milepost Place #606
Toronto M4H 1C7, Canada
email: iceristeski@hotmail.com

Kostadin G. Trencevski
St. Cyril and Methodius University
Institute of Mathematics
P.O. Box 162, 1000 Skopje, Macedonia
email: kostatre@iunona.pmf.ukim.edu.mk