A PROOF OF JENSEN'S INEQUALITY

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Jensen's inequality is one of the most important inequalities in all of mathematics. The version of this inequality for sums states that for any convex function $\phi(x)$ on an interval (a, b), the inequality

$$\phi(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \leq p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_n\phi(x_n), \tag{1}$$

holds for any positive numbers $p_1, \ldots, p_n$ satisfying $p_1 + \cdots + p_n = 1$, and any numbers $x_1, \ldots, x_n$ in (a, b). Setting $n = 2$ in the above inequality gives the analytic definition of a convex function. Geometrically, convexity asserts that the curve lies on or below the whole chord. The purpose of this note is to present a short and simple proof of Jensen's inequality, both for sums and integrals, which could not be located in the literature by the author.

Jensen's inequality for sums will be proved assuming differentiability of ϕ. Afterwards, Jensen's inequality for integrals will be proved in complete generality. The proof of (1) uses the well–known property of differentiable convex functions that its derivative is an increasing function. Put

$$S = \sum_{i=1}^{n} p_i\phi(x_i) - \phi(A), \quad \text{where } A = \sum_{i=1}^{n} p_ix_i.$$

Jensen's inequality becomes $S \geq 0$. We have

$$S = \sum_{i=1}^{n} p_i\{\phi(x_i) - \phi(A)\} = \sum_{i=1}^{n} p_i \int_A^{x_i} \phi'(x)dx.$$

If $x_i \geq A$, the above integral is bounded below by $\phi'(A)(x_i - A)$. It is easily checked that this bound also holds in the case $x_i < A$. Thus,

$$S \geq \sum_{i=1}^{n} p_i\phi'(A)(x_i - A) = \phi'(A)\left(\sum_{i=1}^{n} p_ix_i - A\right) = \phi'(A)(A - A) = 0.$$

This proof shows, moreover, that assuming $\phi'(x)$ is strictly increasing, equality holds in Jensen's inequality if and only if all the x_i are equal.

The classical arithmetic mean–geometric mean inequality is obtained from Jensen's inequality by taking the convex function to be the exponential function. The above proof specialized to this case is similar to the proof given in [1], though in this proof the property that the derivative of the natural logarithm is decreasing was used instead.

The statement of Jensen's inequality for integrals is taken from [6].

Theorem 1. Let μ be a positive measure on a σ–algebra in a set Ω, so that $\mu(\Omega) = 1$. (That is, μ is a probability measure.) If f is a real–valued integrable function (with respect to μ), $a < f(t) < b$ for all $t \in \Omega$, and if ϕ is convex on (a, b), then

$$\phi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\phi \circ f) d\mu.$$

Note: The cases $a = -\infty$ and $b = \infty$ are not excluded.

In the proof a few simply–derived facts on convex functions are used [4,7]. The left–hand derivative $D^-\phi(x)$ and the right–hand derivative $D^+\phi(x)$ exist for all $x \in (a, b)$. If $a < x < y < b$, then

$$D^+\phi(x) \leq \frac{\phi(y) - \phi(x)}{y - x} \leq D^-\phi(y) \leq D^+\phi(y). \tag{2}$$

Let $A = \int f d\mu$, and let $S = \int (\phi \circ f) d\mu - \phi(A)$. Jensen's inequality becomes $S \geq 0$. We get by inequality (2)

$$S = \int_\Omega \{(\phi \circ f)(t) - \phi(A)\} d\mu \geq \int_\Omega \{f(t) - A\} D^+\phi(A) d\mu = 0.$$

A last remark: Jensen's inequality for integrals implies (1). Simply take the function f to be the identity, and take the measure μ to be $\sum_{i=1}^n p_i \delta(x - x_i)$, where $\delta(x)$ is the Dirac delta function.

References

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