COMMUTATIVITY MODULO RADICAL AND SPECTRA IN L. M. C. - ALGEBRAS

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Abstract. A locally multiplicative convex algebra E is commutative modulo the radical $R(E)$ if and only if the left joint spectrum of a finite set of elements is contained in the right joint spectrum of these elements.

1. Introduction. Let E be a commutative complete l.m.c. algebra and let $x_1, x_2, \dots, x_n$ be in E. Then the joint spectrum [1, 3] of $x_1, x_2, \dots, x_n$ is the subset of $\mathbb{C}$, the complex plane, defined by:

$$sp(E; x_1, x_2, \dots, x_n) = \{(\phi(x_1), \phi(x_2), \dots, \phi(x_n)) \ : \ \phi \in \triangle(E)\},$$

where $\triangle(E)$ is the maximal ideal space of E. In other words, $sp(E; x_1, x_2, \dots, x_n)$ is the set of complex n-tuples $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that either $E(\lambda_1 - x_1) + \cdots + E(\lambda_n - x_n)$ is a proper left ideal or $(\lambda_1 - x_1)E + \cdots + (\lambda_n - x_n)E$ is a proper right ideal. If $n = 1$, then the joint spectrum of a single element reduces to the usual notion of spectrum.

Let E be a complete l.m.c. algebra with unit and with a fixed family $\{p_\alpha\}$, $\alpha \in I$ of submultiplicative seminorms. For each $\alpha \in I$, let E_α be the Banach algebra with unit and $q_\alpha \colon E \to E_\alpha$ be the quotient map, then E is imbedded as a dense subalgebra of the projective limit via the Arens-Michael decomposition. Denote by $\Omega(E)$ the set of all spectral states of E and $\rho_\alpha(x)$ is the spectral radius of $x_\alpha \in E_\alpha$. Then,

$$\Omega(E) = \bigcup_\alpha q_\alpha^*(\Omega(E_\alpha)),$$

where q_α^* is the adjoint of q_α and

$$\Omega(E_\alpha) = \{f \in E_\alpha^* \ : \ f(e) = 1, \ |f(x)| \le \rho_\alpha(x), \ x \in E_\alpha\},$$

(see [1]). Let $\triangle_\alpha$ be the set of all multiplicative linear functionals on E_α and let $R(E)$ be the radical of E.

Theorem 1.1. Let E be a complete l.m.c. algebra with unit. Then E is commutative modulo $R(E)$ if and only if $co\ sp(E; x) = \{f(x) \ : \ f \in \Omega(E)\}$.

Proof. Let E be commutative modulo $R(E)$. Then for $x, y, z \in E$, $z(xy - yx)$ is quasi-regular [3].

For each $\alpha \in E$,

$$\rho_{E_\alpha}(x_\alpha y_\alpha - y_\alpha x_\alpha) = 0,$$

because

$$\rho_E(xy - yx) = 0 \text{ and } \rho_E = \sup_\alpha \rho_{E_\alpha}.$$

This proves that E_α is commutative modulo $R(E_\alpha)$, [4]. So by the definition of Ω_α we obtain

$$sp(E_\alpha; x_\alpha) = \{\phi_\alpha(x_\alpha) \ : \ \phi_\alpha \in \triangle_\alpha\}.$$

Since $\triangle_\alpha \subset \Omega_\alpha$ and Ω_α is a convex set, we also have

$$co\ sp(E_\alpha; x_\alpha) \subset \{f_\alpha(x_\alpha) \ : \ f_\alpha \in \Omega_\alpha\}.$$

Further, the family of spectra is directed and

$$sp(E; x) = \bigcup_\alpha co\ sp(E_\alpha; x_\alpha),$$

which also proves that

$$co\ sp(E_\alpha; x_\alpha) = \{f_\alpha(x_\alpha) \ : \ f_\alpha \in \Omega_\alpha\}.$$

This, in turn, implies that

$$co\ sp(E; x) = \{f(x) \ : \ f \in \Omega_E\}.$$

We will show that the condition

$$co\ sp(E; x) = \{f(x) \ : \ f \in \Omega_E\}$$

implies that E is commutative modulo $R(E)$.

Let $f \in \Omega(E)$. Then for $x, y \in E$, $sp(E; (xy - yx)) = \{0\}$. It is obvious that E_α is commutative modulo $R(E_\alpha)$, $\alpha \in I$. Hence, for $x, y, z \in E$, we have $z(xy - yx)$ is quasi-regular in E. This shows that $xy - yx \in R(E)$ and thus, E is commutative modulo $R(E)$.

2. Joint Spectra and R(E). In this section we establish a necessary and a sufficient condition for an l.m.c. algebra E to be commutative modulo $R(E)$. We denote the left and right joint spectra of E by $SPL(E)$ and $SPR(E)$, respectively. These spectra are defined as follows.

$$SPL(E) = \left\{ \lambda_i \in \mathbb{C} \ : \ \sum_{i=1}^{n} E(x_i - \lambda_i) \neq E \right\}$$

$$SPR(E) = \left\{ \lambda_i \in \mathbb{C} \ : \ \sum_{i=1}^{n} (x_i - \lambda_i)E \neq E \right\}.$$

We prove the following theorem.

Theorem 2.1. Let E be a complete complex l.m.c. algebra with unit and let $x_1, x_2, \dots, x_n$ be in E. Then $SPL(E) \subset SPR(E)$ if and only if E is commutative modulo radical $R(E)$.

Proof. Let E_α be a commutative modulo $R(E_\alpha)$ and

$$x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n} \in E_\alpha.$$

Then

$$sp(E_\alpha / R(E_\alpha)) = SPL(E_\alpha) = SPR(E_\alpha).$$

If $M \in \triangle(E_\alpha)$, then

$$ME_\alpha = \left\{ \sum_{i=1}^{k} m_i x_{\alpha_i} \ : \ m_i \in M, \ x_{\alpha_i} \in E_\alpha, \ i = 1, \dots, k; \ k = 1, 2, \dots \right\}$$

and $M \subset ME_\alpha$. Suppose $M \neq ME_\alpha$. Then there exist $m_1, m_2, \dots, m_n \in M$ and

$$x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n} \in E_\alpha$$

such that

$$x_\alpha = m_1 x_{\alpha_1} + \cdots + m_n x_{\alpha_n} \neq M.$$

If M_l is the left ideal of E_α, then the maximality of M implies that $M_l = E_\alpha$. This shows that

$$e_\alpha = m + \overline{m}_1 x_{\alpha_1} + \cdots + \overline{m}_n x_{\alpha_n},$$

where $\overline{m}_i = ym_i \in M$, $i = 1, 2, \ldots, n$, and e_α is the unit of E_α. This proves that

$$0 \notin SPR(m, \overline{m}_1, \ldots, \overline{m}_n)$$

and hence, there exists $\gamma_0, \gamma_1, \ldots, \gamma_n \in E_\alpha$ such that

$$\gamma_0 m + \gamma_1 \overline{m}_1 + \cdots + \gamma_n \overline{m}_n = e_\alpha.$$

Thus, $e_\alpha \in M$, which contradicts the fact that M is a proper ideal. So we have shown that M is a two-sided ideal and $M = ME_\alpha$. Let E_α / M be the quotient algebra. If H_α is the set of all nonzero elements of this quotient algebra, then any element in H_α has a left inverse and H_α is a group as well as a semigroup. By the Gelfand-Mazur Theorem we have $E_\alpha / M \cong \mathbb{C}$. Hence, $E_\alpha / R(E_\alpha)$ is commutative.

The following definition comes from [2].

Definition 2.1. The left and right approximate point spectra of

$$x_{\alpha_1}, \ldots, x_{\alpha_n} \in E_\alpha$$

are defined as follows.

$$SPL_P(x_{\alpha_1}, \ldots, x_{\alpha_n}) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \ : \ \inf_{\|y_\alpha\|_\alpha = 1} \sum_{i=1}^{n} \|(x_{\alpha_i} - \lambda_i) y_\alpha\| = 0 \right\}$$

$$SPR_P(x_{\alpha_1}, \ldots, x_{\alpha_n}) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \ : \ \inf_{\|y_\alpha\|_\alpha = 1} \sum_{i=1}^{n} \|y_\alpha (x_{\alpha_i} - \lambda_i)\| = 0 \right\},$$

respectively. Also, note that

$$SP_P = SPL_P \bigcup SPR_P,$$

where SP_P denotes the point spectrum.

Let $SPL_P(x)$ and $SPR_P(x)$ denote the left and the right approximate point spectra of x in E.

Theorem 2.2. For each $\alpha \in I$ and $x_\alpha \in E_\alpha$,

$$SPL_P(x_\alpha) \subset SPR_P(x_\alpha) \text{ implies } SPL(x_\alpha) = SPR(x_\alpha).$$

Proof. By the definition of spectrum, the condition $SPL(x_\alpha) = SPR(x_\alpha)$ is equivalent to the following condition.

If for $x_\alpha, y_\alpha \in E_\alpha$ and $x_\alpha y_\alpha = e_\alpha$, then $y_\alpha x_\alpha = e_\alpha$, where e_α is the identity in E_α, [1]. Let $SPL_P(x_\alpha) \subset SPR_P(x_\alpha)$ and $x_\alpha y_\alpha = e_\alpha$. Then for $\alpha \in I$ and $u_\alpha \in E_\alpha$,

$$\|u_\alpha x_\alpha y_\alpha\|_\alpha = \|u_\alpha\|_\alpha \leq \|u_\alpha x_\alpha\|_\alpha \|y_\alpha\|_\alpha.$$

This shows that $0 \notin SPR_P(x_\alpha)$ and since $SPL_P(x_\alpha) \subset SPR_P(x_\alpha)$, we have $0 \notin SPL_P(x_\alpha)$.

Hence, there exists a $\epsilon > 0$ such that $\epsilon\|x_\alpha\|_\alpha \leq \|x_\alpha u_\alpha\|_\alpha$ for all $x_\alpha \in E_\alpha$. Let $z_\alpha = y_\alpha x_\alpha$. Then

$$x_\alpha z_\alpha = x_\alpha(y_\alpha x_\alpha) = (x_\alpha y_\alpha)x_\alpha = e_\alpha x_\alpha.$$

That is, $0 = x_\alpha(e_\alpha - z_\alpha)$ and

$$0 = \|x_\alpha(e_\alpha - z_\alpha)\|_\alpha \geq \epsilon\|e_\alpha - z_\alpha\|_\alpha.$$

Thus, $e_\alpha = z_\alpha$ and the theorem follows.

Corollary 2.3.

$$SPL_P(x) \subset SPR_P(x) \text{ implies } SPL(x) = SPR(x),$$

for all $x \in E$.

Proof. Since

$$sp(E; x) = \bigcup_\alpha sp(E_\alpha; x_\alpha),$$

the corollary follows from Theorem 2.2 and the fact that the union of left and right joint approximate spectra is the joint spectrum and the union of left and right approximate point spectra is the joint approximate point spectrum [2].

Example 2.1. Let X be the vector space of the 3×3 matrices generated by the following matrices and the identity matrix I_3.

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ X_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Of all possible products of these matrices, the nonzero products are

$$X_2^2 = X_1,\ X_4^2 = X_4,\ X_1X_4 = X_1,\ X_3X_4 = X_3 = X_2X_1.$$

Hence, X is an algebra generated by X_1, X_2, X_3, X_4 and I_3. An arbitrary element x of X has the following form.

$$x = \alpha I_3 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4,$$

where α_i are scalars. Let ϕ_1, ϕ_2, and ϕ_3 be the multiplicative functionals on X. Then $\phi_1(x) = \alpha$, $\phi_2(x) = \alpha + \alpha_2$, and $\phi_3(x) = \alpha + \alpha_4$ and

$$R(X) = \bigcap \{ \text{ the kernels of } \phi_i,\ i = 1, 2, 3\}.$$

Hence, by Theorem 2.1, $X/R(X)$ is commutative.

Further,

$$SPL_P(X; I_3 + X_1; X_2) = \{(1,0),\ (1,1)\} \not\subset SPL_R(X, I_3 + X_1; X_2) = \{(1,0)\}.$$

The above example estalishes that there are algebras where $X/R(X)$ is commutative but $SPL_P \not\subset SPL_R$.

Last but not least, we have the following question.

Does the condition $SPL_P \subset SPL_R$ imply that $X/R(X)$ is commutative?

We note that the last example gives a converse of the above posed question.

References

1. F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras,* Vol. I, London Math. Soc., Lect. Notes 2, Cambridge University Press, Cambridge, 1971.
2. R. E. Harte, "Spectral Mapping Theorems," *Proc. Roy. Irish Acad. Sect. A,* 72 (1972), 89–107.

3. R. Larsen, *Banach Algebras An Introduction,* Marcel Dekker, Inc., New York, 1973.

4. C. Le Page, "Sur Quelques Conditions Entrainant la Commutative dans les Algebras de Banach," *C. R. Acad. Sc. Paris,* A 265 (1967), 235–237.

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