THE CONGRUENT-INCIRCLE CEVIANS OF A TRIANGLE

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Abstract. A congruent-incircle cevian of a triangle subdivides the triangle into two triangles with congruent incircles. We give two Euclidean constructions of such cevians, and also construct Heronian triangles for which the three pairs of congruent-incircle subtriangles are all Heronian.

1. Introduction. Consider the problem of dividing a given triangle into two subtriangles with congruent incircles: given a triangle ABC, find a Euclidean construction of the point P on the side BC so that the subtriangles ABP and ACP have congruent incircles. We shall refer to these subtriangles as the congruent-incircle subtriangles on the side BC, and AP the corresponding congruent-incircle cevian of the triangle. The other two congruent-incircle cevians BQ and CR are analogously defined. In Section 6, we study the condition for which the three congruent-incircle cevians are concurrent. We also construct Heronian triangles for which the three pairs of congruent-incircle subtriangles are all Heronian.

2. Preliminaries. Throughout this paper, we shall denote by a, b, c, respectively the lengths of the sides BC, CA, AB of triangle ABC, and α, β, γ the opposite angles of these sides. The area of the triangle and its inradius are given by

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{and} \qquad r = \frac{\triangle}{s},$$

where $s := \frac{1}{2}(a+b+c)$ is the semiperimeter. It is convenient to work with the quantities

$$\tau_1 = \tan\frac{\alpha}{2}, \qquad \tau_2 = \tan\frac{\beta}{2}, \qquad \tau_3 = \tan\frac{\gamma}{2}.$$

These "half-tangents" satisfy the basic relation

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = 1. \tag{1}$$

The triangle ABC is similar to a *standard* triangle with unit semiperimeter, namely, the one with sides

$$a_0 = \tau_1(\tau_2+\tau_3), \qquad b_0 = \tau_2(\tau_3+\tau_1), \qquad c_0 = \tau_3(\tau_1+\tau_2).$$

The area and inradius of this standard triangle are both given by $\tau_1\tau_2\tau_3$. Details can be found in [1, 2].

3. Congruent – Incircle Subtriangles. Consider the congruent-incircle cevian AP of triangle ABC. Suppose this has length x, and divides the side BC in the ratio $BP : PC = k : 1-k$. Equating the inradii of the subtriangles ABP and ACP, we have

$$\frac{2k\triangle}{c+x+ka} = \frac{2(1-k)\triangle}{b+x+(1-k)a}. \tag{2}$$

This equation can be rearranged into the form

$$\frac{c+x}{k} + a = \frac{b+x}{1-k} + a.$$

Here, before cancelling the common term a on both sides, we observe that the resulting equation would be the same even if we change signs and consider

$$\frac{2k\triangle}{c+x-ka} = \frac{2(1-k)\triangle}{b+x-(1-k)a} \tag{3}$$

instead of (2). In either case, we have

$$k : 1-k = x+c : x+b. \tag{4}$$

Now, equation (3) asserts the congruence of the excircles of the subtriangles on the opposite sides of the vertex A. Hence, *the subtriangles ABP and ACP have congruent incircles if and only if they have congruent excircles on the opposite sides of A.* See Figure 1.

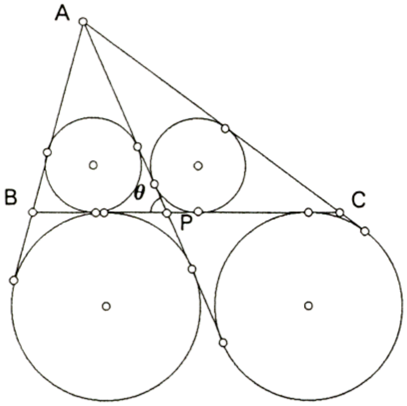

Figure 1.

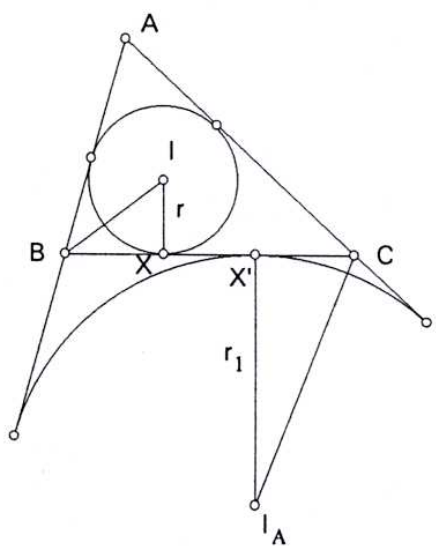

Figure 2.

4. Construction of Congruent – Incircle Cevian. Suppose the incircle touches the side BC at X, and the excircle touches it at X'. It is well known that

$$\begin{aligned} BX &= CX' = s - b, \\ CX &= BX' = s - c. \end{aligned}$$

Denote by r and r_1 the inradius and the radius of the excircle on the side BC. From Figure 2, it is clear that

$$\tau_2 = \frac{r}{s-b} \qquad \text{and} \qquad \tau_3 = \frac{s-b}{r_1}.$$

It follows that

$$\frac{r}{r_1} = \tau_2\tau_3. \tag{5}$$

Consider the congruent-incircle subtriangles ABP and ACP again. Suppose they have common inradius ρ. As noted above, these subtriangles also have congruent excircles on the opposite sides of A, say, of radii ρ_1. Denote by θ the magnitude of angle APB, so that angle APC has magnitude $\pi - \theta$. Applying (5) to the subtriangle ABP, we obtain

$$\frac{\rho}{\rho_1} = \tau_2 \cdot \tan\frac{\theta}{2}.$$

On the other hand, from the subtriangle ACP, we have

$$\frac{\rho}{\rho_1} = \tan\frac{\pi-\theta}{2} \cdot \tau_3 = \frac{\tau_3}{\tan\frac{\theta}{2}}.$$

Combining these two equations, we have

$$\left(\frac{\rho}{\rho_1}\right)^2 = \tau_2\tau_3.$$

From this, we also obtain

$$\tan\frac{\theta}{2} = \sqrt{\frac{\tau_2}{\tau_3}}. \tag{6}$$

In terms of the sides of triangle ABC, we have

$$\tan\frac{\theta}{2} = \sqrt{\frac{s-b}{s-c}} = \frac{\sqrt{(s-b)(s-c)}}{s-c} = \frac{\sqrt{BX\cdot XC}}{XC},$$

where X is the point of contact of the incircle of triangle ABC with the side BC.

This leads to the following construction of the point P.

Suppose the incircle of triangle ABC has center I, and touches the side BC at X.

(1) Extend XI to intersect the semicircle with diameter BC at the point Y.
(2) Construct the line through A parallel to YC, intersecting the line BC at Q.
(3) Construct the perpendicular bisector of AQ to intersect BC at P.

Then AP is the congruent-incircle cevian on the side BC. See Figure 3 below.

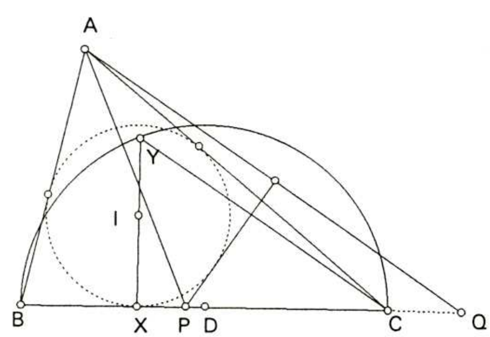

Figure 3.

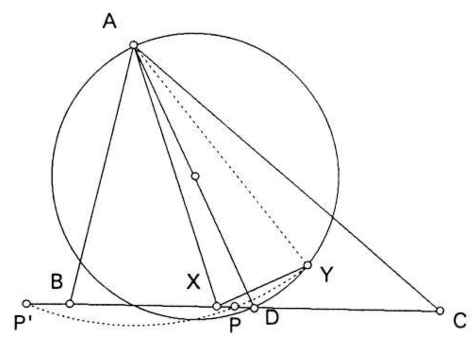

Figure 4.

5. An Alternative Construction.

Proposition 1. The length of congruent-incircle cevian AP is $\sqrt{s(s-a)}$.

Proof. Applying the law of sines to triangle ABP, we have $AP = (c\sin\beta)/(\sin\theta)$. Now, making use of (6), we have

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2\sqrt{(s-b)(s-c)}}{a}.$$

It follows that

$$AP = \frac{ac\sin\beta}{2\sqrt{(s-b)(s-c)}} = \frac{\triangle}{\sqrt{(s-b)(s-c)}} = \sqrt{s(s-a)}$$

by Heron's formula.

Given a triangle ABC, let D be the midpoint of BC, and let the bisector of angle A intersect BC at X. Lau [3] has established an interesting formula which leads to an alternative construction of the congruent-incircle cevian AP.

Lemma 2 (Lau). $s(s-a)$ is equal to the dot product of the median AD and the angle bisector AX.

This means that if the perpendicular from X to AD is extended to intersect the circle with diameter AD at a point Y, then $AY = \sqrt{s(s-a)}$. Now, the circle $A(Y)$ intersects the side BC at two points, one of which is the required point P. See Figure 4.

6. Condition for Concurrency. We consider the condition for the three congruent-incircle cevians of a triangle to be concurrent. It is convenient to replace the triangle by the standard one with unit perimeter. In this case, these cevians have lengths $\sqrt{\tau_2\tau_3}$, $\sqrt{\tau_3\tau_1}$, and $\sqrt{\tau_1\tau_2}$, respectively.

Writing

$$\tau_2\tau_3 = u^2, \qquad \tau_3\tau_1 = v^2, \qquad \tau_1\tau_2 = w^2, \tag{7}$$

for positive numbers u, v, and w, we have

$$BP : PC = u + c_0 : u + b_0 = u + 1 - w^2 : u + 1 - v^2,$$

$$CQ : QA = v + a_0 : v + c_0 = v + 1 - u^2 : v + 1 - w^2,$$

$$AR : RB = w + b_0 : w + a_0 = w + 1 - v^2 : w + 1 - u^2.$$

These cevians AP, BQ, CR are concurrent if and only if

$$BP \cdot CQ \cdot AR = PC \cdot QA \cdot RB.$$

This is equivalent to

$$-(u+1-w^2)(v+1-u^2)(w+1-v^2) + (u+1-v^2)(v+1-w^2)(w+1-u^2) = 0.$$

The polynomial on the left hand side clearly vanishes if any two of u, v, and w are equal. This means that it is divisible by $(u-v)(v-w)(w-u)$. It is then not very hard to figure out the complete factorization so that the condition for concurrency is

$$(u-v)(v-w)(w-u)(1+u+v+w+uv+vw+wu) = 0.$$

Since u, v, and w are all positive, this is zero precisely when two of u, v, w are equal. From this the following proposition is obvious.

Proposition 3. The three congruent-incircle cevians of a triangle are concurrent if and only if the triangle is isosceles.

7. Rationality of Congruent – Incircle Cevian. Now suppose ABC is a Heronian triangle, namely, with integer sides and integer area. We consider the possibility that the congruent-incircle subtriangles ABP and ACP also are Heronian. We shall also construct Heronian triangles for which all three pairs of congruent-incircle subtriangles are Heronian. It is convenient to work with triangles with *rational* sides and *rational* areas. (We shall call such triangles rational). Then one obtains Heronian triangles by magnifying appropriately.

Lemma 4. The similarity class of a triangle contains rational and Heronian triangles if and only if τ_1, τ_2, τ_3 are all rational.

Proposition 5. Let ABC be a rational triangle. The two congruent-incircle subtriangles on the side BC are rational if and only if $\tau_2\tau_3$ is the square of a rational number. Consequently, all three pairs of congruent-incircle subtriangles are rational if and only if $\tau_2\tau_3$, $\tau_3\tau_1$ and $\tau_1\tau_2$ are all rational squares.

Proof. Clearly, a pair of congruent-incircle subtriangles are rational if and only if the corresponding cevian is rational. The first part now follows from (6). The rest is then clear.

Proposition 6. The congruent-incircle subtriangles on the hypotenuse of a right triangle cannot be rational.

While this follows as a corollary of Proposition 5 by analyzing an appropriate Diophantine equation, we give here a short geometric proof, invoking a famous theorem of Fermat.

Proof. It is well known that the inradius of a right triangle is $r = s - c$, where c is the hypotenuse. The length of the congruent-incircle cevian on the hypotenuse, by Proposition 1, is given by $\sqrt{s(s-c)} = \sqrt{rs} = \sqrt{\triangle}$. Fermat has proved that the area of a Pythagorean triangle cannot be a square. This completes the proof of the proposition.

8. Construction of Rational Triangles. To construct a rational triangle with all congruent –circles subtriangles rational, we seek positive *rational* numbers u, v, w satisfying

$$u^2 + v^2 + w^2 = 1. \tag{8}$$

Then with

$$\tau_1 = \frac{vw}{u}, \qquad \tau_2 = \frac{wu}{v}, \qquad \tau_3 = \frac{uv}{w},$$

we have a rational triangle with all three congruent-incircle cevians rational.

The general *positive* rational solution of (8) is of the form

$$u = \frac{t_1^2 - t_2^2 - t_3^2}{t_1^2 + t_2^2 + t_3^2}, \qquad v = \frac{2t_1t_2}{t_1^2 + t_2^2 + t_3^2}, \qquad w = \frac{2t_1t_3}{t_1^2 + t_2^2 + t_3^2}.$$

Here, t_1, t_2, t_3 are positive integers satisfying $t_2^2+t_3^2 < t_1^2$. See, for example, [4]. We may assume t_1, t_2, t_3 are relatively prime. From these facts, we obtain a rational triangle with all congruent-incircle subtriangles rational. To simplify expressions, we magnify by a factor $(t_1^2+t_2^2+t_3^2)^2$, and obtain a Heronian triangle with sides

$$a = (v^2+w^2)(t_1^2+t_2^2+t_3^2)^2 = 4t_1^2(t_2^2+t_3^2),$$

$$b = (w^2+u^2)(t_1^2+t_2^2+t_3^2)^2 = [(t_1-t_2)^2+t_3^2][(t_1+t_2)^2+t_3^2]$$

$$c = (u^2+v^2)(t_1^2+t_2^2+t_3^2)^2 = [(t_1-t_3)^2+t_2^2][(t_1+t_3)^2+t_2^2].$$

The cevians AP, BQ, CR have integer lengths

$$(t_1^2-t_2^2-t_3^2)(t_1^2+t_2^2+t_3^2), \quad 2t_1t_2(t_1^2+t_2^2+t_3^2), \quad 2t_1t_3(t_1^2+t_2^2+t_3^2).$$

These cevians divide the sides of the triangle in the ratios

$$\begin{aligned}
BP : PC &= u+c : u+b \\
&= t_1^2+t_2^2-t_3^2 : t_1^2-t_2^2+t_3^2, \\
CQ : QA &= v+a : v+c \\
&= 2t_1[t_1^2t_2+(2t_1+t_2)(t_2^2+t_3^2)] : (t_1^2+t_1t_3+t_2^2+t_3^2)^2-5t_1^2t_3^2, \\
AR : RB &= w+b : w+a \\
&= (t_1^2+t_1t_2+t_2^2+t_3^2)^2-5t_1^2t_2^2 : 2t_1[t_1^2t_3+(2t_1+t_3)(t_2^2+t_3^2)].
\end{aligned}$$

Note that these segments need not have integer lengths. The rational congruent-incircle subtriangles nevertheless can be made Heronian by further magnification.

The three pairs of congruent incircles have radii

$$2t_2t_3(t_1^2-t_2^2-t_3^2), \quad \frac{4t_1^2t_2t_3(t_1^2-t_2^2-t_3^2)}{(t_1+t_2)^2+t_3^2}, \quad \frac{4t_1^2t_2t_3(t_1^2-t_2^2-t_3^2)}{(t_1+t_3)^2+t_2^2}$$

respectively.

9. A Numerical Example. With $(t_1, t_2, t_3) = (3, 2, 1)$, we obtain

$$(u, v, w) = \left(\frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right), \qquad (\tau_1, \tau_2, \tau_3) = \left(\frac{9}{7}, \frac{1}{7}, \frac{4}{7}\right),$$

leading to the standard rational triangle with sides

$$(a_0, b_0, c_0) = \left(\frac{45}{49}, \frac{13}{49}, \frac{40}{49}\right).$$

Magnifying by a factor $49 \cdot 65$, we obtain the Heronian triangle

$$(a, b, c; \triangle, r) = \left(2925, 845, 2600; 1064700, \frac{2340}{7}\right).$$

The congruent-incircle subtriangles on each side are all Heronian.

side	cevian	subtriangle	subtriangle	inradius
2925=1950+975	910	(2600,1950,910)	(910,975,845)	260
845=435+410	2730	(2925,435,2730)	(2730,410,2600)	180
2600=884+1716	1365	(845,884,1365)	(1365,1716,2925)	234

10. Isosceles Triangles. The triangle constructed in Section 8 is isosceles provided two of the quantities u, v and w are equal. In terms of t_1, t_2, t_3, this condition can be put into the form

$$(t_2 - t_3)(t_1^2 - t_2^2 - t_3^2 - 2t_1t_2)(t_1^2 - t_2^2 - t_3^2 - 2t_1t_3) = 0.$$

If $t_3 = t_2$, and $t_1^2 > 2t_2^2$, writing t for the rational number t_2/t_1, we obtain an isosceles triangle with vertical angle $\alpha = 4\arctan(2t^2)$. By obvious symmetry, the median on the base is a congruent-incircle cevian.

Proposition 7. The congruent-incircle subtriangles on the slant side of a rational isosceles triangle with vertical angle α are rational if and only if $\alpha = 4\arctan(2t^2)$ for a rational number $t < (\sqrt{2})/2$.

The sides of the isosceles triangle can be taken as

$$a = 8t^2, \qquad b = c = 1 + 4t^4.$$

The congruent-incircle cevian BQ on the slant side AC has length $2t(1+2t^2)$, and this divides the slant side into two segments CQ and QA of lengths

$$\frac{2t(1-2t+2t^2)(1+4t+2t^2)}{1+2t+2t^2} \quad \text{and} \quad \frac{(1-2t+2t^2)(1+2t+4t^3+4t^4)}{1+2t+2t^2},$$

respectively. The congruent-incircle subtriangles BCQ and BAQ are both rational, and have areas

$$\frac{8t^3(1-2t^2)(1+2t^2)(1+4t+2t^2)}{(1+2t+2t^2)^2} \text{ and } \frac{4t^2(1-2t^2)(1+2t^2)(1+2t+4t^3+4t^4)}{(1+2t+2t^2)^2}.$$

For example, with $t = \frac{1}{2}$, we obtain an isosceles triangle with vertical angle $4\arctan\frac{1}{2}$. This leads to the standard isosceles rational triangle with sides $\frac{5}{9}, \frac{5}{9}, \frac{8}{9}$. Magnifying by a factor 45, we obtain the isosceles Heronian triangle (25,25,40). Dividing each slant side into two segments of lengths 11 and 14, we obtain the Heronian triangles (25, 11, 30) and (30, 14, 40) each of inradius 4.

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