MEANS AND THEIR ENDS

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The Arithmetic-Geometric Mean Inequality [1] guarantees that

$$GM(a_1, a_2, \ldots, a_n)/AM(a_1, a_2, \ldots, a_n) \leq 1$$

for any finite set of positive terms $\{a_1, a_2, \ldots, a_n\}$. (Here GM and AM denote the geometric and arithmetic means, respectively.) Except in the case where the numerator and denominator are equal, however, we are given no clue as to what the value of that ratio might be. In [2] I showed that for any positive real number s

$$\lim_{n\to\infty} \frac{GM(1^s, 2^s, \ldots, n^s)}{AM(1^s, 2^s, \ldots, n^s)} = \frac{s+1}{e^s}.$$

The special case $s = 1$ leads to the well-known result that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

In fact, a more general result holds for polynomial sequences and also for positive power sequences.

Proposition 1. If $a_k = c_s k^s + c_{s-1}k^{s-1} + \cdots + c_1 k + c_0$ is positive for each k or if $a_k = ck^s$ for c and s positive real numbers, then

$$\lim_{n\to\infty} \frac{GM(a_1, a_2, \ldots, a_n)}{AM(a_1, a_2, \ldots, a_n)} = \frac{s+1}{e^s}. \tag{1}$$

Furthermore, if we define arithmetic and geometric means for a continuous positive function f by

$$AM(f, n) = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x \quad \text{and} \quad GM(f, n) = \exp\left[\frac{1}{n}\int_0^n \ln f(x)\,\mathrm{d}x\right]$$

whenever these integrals are defined [1], then for positive polynomials $f(x) = c_s x^s + c_{s-1}x^{s-1} + \cdots + c_1 x + c_0$, and also for positive power functions $f(x) = cx^s$, we have the continuous version of (1).

Proposition 2.

$$\lim_{n\to\infty} \frac{GM(f,n)}{AM(f,n)} = \frac{s+1}{e^s}. \tag{2}$$

Proof. We prove these results first in the continuous case (2) and then apply a Riemann sum argument to get the discrete case (1).

To prove (2), it suffices to show that

$$\ln GM(f,n) - \ln AM(f,n) \to \ln(s+1) - s. \tag{3}$$

Assume that $f(x) = c_s x^s + c_{s-1}x^{s-1} + \cdots + c_1 x + c_0$ and that $f(x) > 0$ for all $x \geq 0$.

Remark 1. We note that

$$AM(f,n) = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x = \frac{1}{n}\int_0^a f(x)\,\mathrm{d}x + \frac{1}{n}\int_a^n f(x)\,\mathrm{d}x \approx \frac{1}{n}\int_a^n f(x)\,\mathrm{d}x$$

and

$$\ln GM(f,n) = \frac{1}{n}\int_0^n \ln f(x)\,\mathrm{d}x = \frac{1}{n}\int_0^a \ln f(x)\,\mathrm{d}x + \frac{1}{n}\int_a^n \ln f(x)\,\mathrm{d}x$$

$$\approx \frac{1}{n}\int_a^n \ln f(x)\,\mathrm{d}x$$

for large $n \gg a$. This allows us to start our integrals wherever we like and thus avoid any potential problems with convergence of improper integrals.

Write

$$f(x) = c_s x^s \left[1 + \frac{c_{s-1}}{c_s}\frac{1}{x} + \cdots + \frac{c_0}{c_s}\frac{1}{x^s}\right] = c_s x^s[1 + g(x)],$$

where $g(x)$ is $O(1/x)$. Then

$$\ln GM(f,n) \approx \frac{1}{n}\int_a^n \ln f(x)\,\mathrm{d}x = \frac{1}{n}\int_a^n [\ln c_s + s\ln x + \ln(1+g(x))]\,\mathrm{d}x$$

$$= \frac{n-a}{n}\ln c_s + \frac{s}{n}[n\ln n - n - a\ln a + a] + \frac{1}{n}\int_a^n \ln(1+g(x))\,\mathrm{d}x.$$

Since $g(x)$ is $O(1/x)$, the last term, being nearly the *average* of $\ln(1+g(x))$ over the interval $[a,n]$, will be small for large n and thus,

$$\ln GM(f,n) \approx \ln c_s + s\ln n - s.$$

On the other hand, if F is the antiderivative of f with $F(0)=0$,

$$\ln AM(f,n) \approx \ln\left[\frac{1}{n}\int_a^n f(x)\,\mathrm{d}x\right] = \ln\left[\frac{c_s n^s}{s+1} + \frac{c_{s-1}n^{s-1}}{s} + \cdots + \frac{c_0}{1} - \frac{F(a)}{n}\right]$$

$$= \ln\left[\left(\frac{c_s n^s}{s+1}\right)\left(1 + \frac{c_{s-1}}{c_s}\frac{s+1}{s}\frac{1}{n} + \cdots + \frac{c_0}{c_s}\frac{s+1}{1}\frac{1}{n^s} - \frac{F(a)}{c_s}\frac{s+1}{1}\frac{1}{n^{s+1}}\right)\right]$$

$$= \ln c_s + s\ln n - \ln(s+1)$$

$$+ \ln\left[1 + \frac{c_{s-1}}{c_s}\frac{s+1}{s}\frac{1}{n} + \cdots + \frac{c_0}{c_s}\frac{s+1}{1}\frac{1}{n^s} - \frac{F(a)}{c_s}\frac{s+1}{1}\frac{1}{n^{s+1}}\right]$$

$$\approx \ln c_s + s\ln n - \ln(s+1),$$

when n is large. Putting these two results together, we have (3), and thus, Proposition 2 is proven.

We will use a Riemann sum argument for the proof of (1). For a positive, increasing function h, a rough sketch would establish the basic sum-versus-integral inequalities that we need.

$$\int_c^n h(x)\,\mathrm{d}x + h(c) \le \sum_{k=c}^{n} h(k) \le \int_c^n h(x)\,\mathrm{d}x + h(n). \tag{4}$$

So let us assume that $f(x) = c_s x^s + c_{s-1}x^{s-1} + \cdots + c_1 x + c_0$, as above, is increasing in addition to being positive and that $\ln f(x)$ is positive for $x \geq 1$. (Certainly these facts are at least *eventually* true, since $c_s > 0$. But just as the lower limit of integration could be freely chosen in the continuous case above (see Remark 1), the choice of a starting point for our sequence does not affect the limits we are interested in. That is, in (1) it does not matter whether we start at $a_1 = f(1)$ or $a_{24} = f(24)$, or wherever.)

Now if $f(x)$ is increasing on $[1, \infty)$, $\ln f(x)$ will be increasing there as well, and applying (4) successively to $\ln f$ and to f yields

$$\frac{1}{n}\int_1^n \ln f(x)\,\mathrm{d}x + \frac{1}{n}\ln a_1 \leq \frac{1}{n}\sum_{k=1}^n \ln a_k \leq \frac{1}{n}\int_1^n \ln f(x)\,\mathrm{d}x + \frac{1}{n}\ln a_n$$

$$\frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_1 \leq \frac{1}{n}\sum_{k=1}^n a_k \leq \frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_n$$

$$\ln\left[\frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_1\right] \leq \ln\left[\frac{1}{n}\sum_{k=1}^n a_k\right] \leq \ln\left[\frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_n\right],$$

the third line following from the monotonicity of the logarithm. Then

$$\begin{aligned}
&\frac{1}{n}\int_1^n \ln f(x)\,\mathrm{d}x + \frac{1}{n}\ln a_1 - \ln\left[\frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_n\right] \\
&\leq \frac{1}{n}\sum_{k=1}^n \ln a_k - \ln\left[\frac{1}{n}\sum_{k=1}^n a_k\right] \\
&\leq \frac{1}{n}\int_1^n \ln f(x)\,\mathrm{d}x + \frac{1}{n}\ln a_n - \ln\left[\frac{1}{n}\int_1^n f(x)\,\mathrm{d}x + \frac{1}{n}a_1\right].
\end{aligned}$$

Exponentiating, we get

$$\frac{GM'(f,n)a_1^{1/n}}{AM'(f,n) + \frac{1}{n}a_n} \leq \frac{GM(a_1, a_2, \ldots, a_n)}{AM(a_1, a_2, \ldots, a_n)} \leq \frac{GM'(f,n)a_n^{1/n}}{AM'(f,n) + \frac{1}{n}a_1}. \tag{5}$$

(Here we denote by GM' and AM' the respective means starting from $x = 1$ rather than $x = 0$.)

Since $AM'(f, n)$ is dominated by $(c_s n^s)/(s+1)$, while a_n/n is dominated by $c_s n^{s-1}$,

$$\frac{AM'(f,n) + \frac{1}{n}a_n}{AM'(f,n)} \to 1.$$

Also, $(\ln a_n)/n \to 0$, so $a_n^{1/n} \to 1$. Moreover, Remark 1 guarantees that

$$\frac{GM'}{GM} \to 1 \quad \text{and} \quad \frac{AM'}{AM} \to 1.$$

And certainly $a_1^{1/n} \to 1$ and $a_1/n \to 0$. As a result of all these facts, both sides of (5) go to $(s+1)/e^s$ by (2) and we are done with the proof of Proposition 1.

Remark 2. One can ask what

$$\lim \frac{GM}{AM}$$

looks like for functions other than polynomials and powers of x. For functions much bigger than powers of x and polynomials, such as $f(x) = e^x$, the limit is 0. For functions much smaller, such as $f(x) = \ln x$, the limit is 1. Certainly for any (asymptotically) constant function, the limit is also 1, since if $f(x) \equiv c$, we have $GM = c = AM$. (Note that this is consistent with the $s = 0$ case of (1) and (2).)

While we have used the expressions "much bigger than polynomials" and "much smaller than polynomials," we should note that the limits discussed here are sensitive to more than just the order of the underlying function. For example, the function $f(x) = (2+\sin x)x^2$ does not have $\lim GM/AM = 3/e^2$ even though it is "of the order of x^2," lying between x^2 and $3x^2$, both of which do have the stated limit.

Remark 3. One might also look at other means, for example, the harmonic means [1].

$$HM(a_1, a_2, \ldots, a_n) = \left[\frac{1}{n}\sum_{k=1}^{n} a_k^{-1}\right]^{-1} \quad \text{and} \quad HM(f,n) = \left[\frac{1}{n}\int_0^n [f(x)]^{-1}\,\mathrm{d}x\right]^{-1}.$$

The harmonic mean is smaller than the geometric mean [1], and for polynomials, powers of x and e^x, $HM/GM \to 0$. If $f(x) \approx c$, then $HM \approx GM \approx AM \approx c$, so $HM/GM \to 1$ and $HM/AM \to 1$. In fact, $f(x) = \ln x$ is small enough so that $HM/AM \to 1$ and, therefore, $HM/GM \to 1$.

Remark 4. Finally, for more general arithmetic means defined by

$$AM_r(a_1, a_2, \ldots, a_n) = \left[\frac{1}{n}\sum_{k=1}^{n} a_k^r\right]^{1/r} \quad \text{and} \quad AM_r(f, n) = \left[\frac{1}{n}\int_0^n [f(x)]^r \,\mathrm{d}x\right]^{1/r},$$

[1], we can show, with only slight modifications to our proofs, that the limits of GM/AM_r are $((rs+1)^{1/r})/(e^s)$ for positive polynomial sequences and functions of degree s. In fact, for such sequences and functions, these limits are special cases of

$$\lim_{n\to\infty} \frac{AM_t}{AM_r} = \frac{(rs+1)^{1/r}}{(ts+1)^{1/t}},$$

since $GM = \lim_{t\to 0} AM_t$ [1].

References

1. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities,* Cambridge, 1983.

2. R. P. Kubelka, "Means to an End," *Mathematics Magazine,* 74 (2001), 141–142.

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